

## PSEUDOINVERSION OF DEGENERATE METRICS

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Let  $(M, g)$  be a smooth manifold  $M$  endowed with a metric  $g$ . A large class of differential operators in differential geometry is intrinsically defined by means of the dual metric  $g^*$  on the dual bundle  $TM^*$  of 1-forms on  $M$ . If the metric  $g$  is (semi)-Riemannian, the metric  $g^*$  is just the inverse of  $g$ . This paper studies the definition of the above-mentioned geometric differential operators in the case of manifolds endowed with degenerate metrics for which  $g^*$  is not defined. We apply the theoretical results to Laplacian-type operator on a lightlike hypersurface to deduce a Takahashi-like theorem (Takahashi (1966)) for lightlike hypersurfaces in Lorentzian space  $\mathbb{R}_1^{n+2}$ .

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**1. Introduction.** The introduction of a Riemannian metric on a smooth manifold gives rise to a series of differential operators which reveal deep relationships between the geometry and the topology of the manifold. If  $\Delta^k$  is the Laplacian on  $k$ -forms, it is well known that, on a compact manifold, its spectrum  $\{\lambda_i^k\}$  contains topological and geometric information on the manifold. According to the Hodge decomposition theorem, the dimension of the kernel of  $\Delta^k$  equals the  $k$ th Betti number so that the Laplacian determines Euler characteristic  $\chi(M)$  of compact Riemannian manifolds  $(M, g)$ .

Among usual differential operators on  $(M, g)$ , the exterior derivative which takes  $k$ -forms to  $(k+1)$ -forms is the only one defined in terms of the smooth structure of the manifold  $M$ . The others are defined by means of the metric  $g^*$  on the dual bundle, which is the inverse of the given metric  $g$ , in semi-Riemannian case.

Many situations arise in mathematical physics where the metric is degenerated and it is not possible to define the inverse  $g^*$ . A typical case is the one we have in the coupling of Einstein theory of gravity with both a quantum mechanics particle with spin and an electromagnetic field, that is, Einstein-Dirac-Maxwell (EDM) system. If, for instance, we consider Lorentz framed manifold  $\bar{M}$  endowed with Eddington-Finkelstein metric (cf. Hawking and Ellis [12, page 150])

$$ds^2 = -\left(1 - \frac{2m}{r}\right) du^2 + 2dudr + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (1.1)$$

given in a coordinate system  $(u, r, \theta, \phi)$ , where  $u = t + \bar{r}$  is an advanced null coordinate with

$$\bar{r} = \int \frac{dr}{1 - 2m/r} = r + 2m \ln(r - 2m) \quad (r > 2m), \tag{1.2}$$

the hypersurface  $M : u = \text{constant}$  is a degenerate hypersurface. Dirac operator can be written outside  $M$  and in its interior region. But it is not easy to match the two operators on  $M$  due to the fact that the inverse metric  $g^*$  cannot be defined on  $M$ . Normalization problems and the geometry of those manifolds are considered in several papers (see [1, 2, 3, 5, 6, 7, 8, 10, 11, 13, 14] and references therein).

In the following, we consider a smooth (semi)-Riemannian manifold  $(M, g)$ , its Levi-Civita connection  $\nabla$ , its tangent (resp., cotangent) bundle  $TM$  (resp.,  $TM^*$ ), and  $\mathcal{F}(M)$  the space of smooth functions on  $M$ . For a vector bundle  $E$ , we denote by  $\Gamma(E)$  the space of its smooth sections. For a smooth function  $f$  in  $\mathcal{F}(M)$ , the gradient of  $f$  is the vector field  $\text{grad}^\theta f$  given by

$$g(\text{grad}^\theta f, X) = X(f) = df(X) \tag{1.3}$$

for any  $X$  in  $\Gamma(TM)$ .

In local coordinates  $(x^1, \dots, x^n)$ , the gradient of  $f$  is given by

$$\text{grad}^\theta f = g^{ij} f_i E_j, \tag{1.4}$$

where  $f_i = \partial f / \partial x^i$ ,  $E_j = \partial / \partial x^j$ , and  $g^{ij}$  is the  $(i, j)$ -entry of the inverse  $g^{-1}$  of  $g$ .

The gradient can be defined using exterior differential  $d$  and the natural isomorphism  $\#$  between the tangent bundle  $TM$  and its dual bundle  $TM^*$ , by the composition

$$\text{grad} : \mathcal{F}(M) \xrightarrow{d} TM^* \xrightarrow{\#} TM, \tag{1.5}$$

that is,

$$\text{grad} = \# \circ d, \tag{1.6}$$

where

$$\begin{aligned} \# : TM^* &\longrightarrow TM, \\ \omega &\longmapsto \omega^\# \end{aligned} \tag{1.7}$$

is such that, for all  $Y \in \Gamma(TM)$ ,

$$g(\omega^\#, Y) = \omega(Y). \tag{1.8}$$

The inverse of # is the isomorphism  $b = (\#)^{-1}$  defined by

$$\begin{aligned} b : TM &\rightarrow TM^*, \\ X &\mapsto X^b \end{aligned} \tag{1.9}$$

such that

$$X^b(Y) = g(X, Y) \tag{1.10}$$

for all  $Y$  in  $\Gamma(TM)$ .

The divergence of a vector field  $X$  is the codifferential of the dual 1-form  $X^b$  defined by

$$\operatorname{div} X = \delta(X^b) = -\operatorname{trace}(Z \mapsto \nabla_Z X). \tag{1.11}$$

The covariant differentiation  $\nabla$  defined on vector fields can be extended to 1-forms by duality using isomorphism #:

$$\nabla \alpha(X, Y) = g(\nabla_X \alpha^\#, Y) \quad \forall \alpha \in T^*M. \tag{1.12}$$

The Laplace-Beltrami operator  $\Delta^g$ , whose physical and mathematical importance is well known, is defined on smooth functions by

$$\Delta^g = \operatorname{div}(\operatorname{grad}) = \delta \circ b \circ \# \circ d, \tag{1.13}$$

that is, locally,

$$\Delta_g f = \sum_{i,j} g^{ij} \left\{ \frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial f}{\partial x^k} \right\}. \tag{1.14}$$

From (1.4) and (1.14), it is obvious that the entries  $g^{ij}$  of the inverse of  $g_{ij}$  need to be computed in local coordinates. So the following question seems to be coming out in a natural way: how can we define those differential operators on a lightlike manifold, that is, on a submanifold with a degenerate (noninvertible) metric  $g$ ?

This question will be answered by [Proposition 3.1](#).

First, note that  $(dx^i)$  and  $(\partial_i^b)$  are two local bases of  $T^*M$  and the decomposition

$$\partial_i^b = \lambda_{ij} dx^j, \quad 1 \leq i \leq n. \tag{1.15}$$

holds.

Hence,

$$g_{ik} = g(\partial_i, \partial_k) = \partial_i^b(\partial_k) = \lambda_{ij} dx^j(\partial_k) = \lambda_{ik}, \tag{1.16}$$

and then,

$$\partial_i^b = g_{ik} dx^k, \quad 1 \leq i \leq n. \quad (1.17)$$

So inverting  $g$ , we get

$$dx^k = g^{ik} \partial_i^b, \quad 1 \leq k \leq n. \quad (1.18)$$

Thus, coefficients  $g^{ij}$  appear as the entries of the matrix of transition from basis  $(\partial_i^b)$  to  $(dx^k)$ .

Then, an important step in answering the above question should be the construction of the isomorphisms  $\#$  and  $\flat$  adapted to the case of lightlike manifolds.

Below, we discuss lightlike submanifolds of semi-Riemannian manifolds. We are particularly interested in lightlike hypersurfaces. One may proceed similarly in any other codimension. The paper is organized as follows. [Section 2](#) deals with preliminaries and specifies our notations and some basic results. [Section 3](#) is devoted to the definition and construction of the pseudoinverse of a degenerate metric  $g$ , and a concept of *associate metric* with the degenerate metric  $g$  is also introduced. In [Section 4](#), the compatibility of this associate metric with respect to the connection induced by  $g$  is examined. The existence and the properties of the pseudoinverse allow the definition of a Laplacian  $\Delta^g$  on smooth functions of  $(M, g)$ . Actually, this  $\Delta^g$  is a d'Alembertian since  $g$  is nondefinite. However, for the sake of simplicity, we call it Laplacian and we denote it by  $\Delta^g$ . It is formally defined in the same way as the Laplace operator in [Section 5](#). The Laplacian of lightlike hypersurfaces endowed with parallel screen distribution is then defined and its action on position vector field is expressed (see [Theorem 5.2](#)). Finally, in [Section 6](#), we give an application which oddly reminds us of Takahashi theorem [15]. It is our [Theorem 6.1](#).

**2. Preliminaries on lightlike hypersurfaces.** Let  $M$  be a hypersurface of an  $(n + 2)$ -dimensional semi-Riemannian manifold  $(\bar{M}, \bar{g})$ . In the classical theory of nondegenerate hypersurface, the normal bundle has trivial intersection  $\{0\}$  with the tangent one and plays an important role in the introduction of main geometric objects. In case of degenerate hypersurface, the situation is totally different. The normal bundle  $TM^\perp$  is a rank-one distribution over  $M : TM^\perp \subset TM$ ; hence, the metric  $g$  on  $M$  is degenerate, that is, noninvertible and of rank  $n$ . In fact, the following propositions is proved in [11]

**PROPOSITION 2.1.** *Let  $(M, g)$  be a hypersurface of an  $(n + 2)$ -dimensional semi-Riemannian manifold  $(\bar{M}, \bar{g})$ . Then the following assertions are equivalent:*

- (i)  $M$  is a lightlike hypersurface of  $\bar{M}$ ;
- (ii)  $g$  has a constant rank  $n$  on  $M$ ;
- (iii)  $TM^\perp = \cup_{x \in M} T_x M^\perp$  is a distribution on  $M$ .

A complementary bundle of  $TM^\perp$  in  $TM$  is a rank- $n$  nondegenerate distribution over  $M$ . It is called a screen distribution in the tangent bundle and is often denoted by  $S(TM)$ . In [11, Chapter 4], degenerate hypersurface  $(M, g)$  endowed with this specific distribution is denoted by  $(M, g, S(TM))$  and is also called lightlike hypersurface. For the convenience of the reader, we summarize the properties of lightlike hypersurfaces which are appropriate to our purpose.

There exists a vector bundle  $\text{tr}(TM)$  of rank one over  $M$  such that for any nonzero section  $\xi$  of  $TM^\perp$  on a coordinate neighbourhood  $\mathcal{U} \subset M$ , there is a unique section  $N$  of  $\text{tr}(TM)$  defined on  $\mathcal{U}$  such that

$$\bar{g}(\xi, N) = 1, \quad \bar{g}(N, N) = \bar{g}(N, W) = 0 \quad \forall W \in \Gamma(S(TM)|_{\mathcal{U}}). \tag{2.1}$$

Let  $\bar{\nabla}$  and  $\nabla$  be the Levi-Civita connection of  $(\bar{M}, \bar{g})$  and an induced connection on  $(M, g, S(TM))$ , respectively. It is known that  $\nabla$  is not unique. It depends on both  $g$  and the considered screen distribution.

With the decompositions into orthogonal direct sums

$$TM = S(TM) \perp TM^\perp, \tag{2.2}$$

$$T\bar{M}|_M = S(TM) \perp (TM^\perp \oplus \text{tr}(TM)) = TM \oplus \text{tr}(TM), \tag{2.3}$$

we have Gauss and Weingarten formulae in the form

$$\begin{aligned} \bar{\nabla}_X Y &= \nabla_X Y + h(X, Y) \quad \forall X, Y \in \Gamma(TM), \\ \bar{\nabla}_V X &= -A_V X + \nabla_X^t V \quad \forall X \in \Gamma(TM), \forall V \in \Gamma(\text{tr}(TM)), \end{aligned} \tag{2.4}$$

or equivalently,

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \tag{2.5}$$

$$\bar{\nabla}_X N = -A_N X + \tau(X)N, \tag{2.6}$$

where we put

$$B(X, Y) = \bar{g}(h(X, Y), \xi) \quad \forall X, Y \in \Gamma(TM|_{\mathcal{U}}), \tag{2.7}$$

$$\tau(X) = \bar{g}(\nabla_X^t N, \xi) \quad \forall X \in \Gamma(TM|_{\mathcal{U}}). \tag{2.8}$$

For  $\nabla_X Y$  and  $A_V X$  belonging to  $\Gamma(TM)$  and  $h$  a  $\Gamma(\text{tr}(TM))$ -valued symmetric  $\mathcal{F}(M)$ -bilinear form on  $\Gamma(TM)$ ,  $A_V$  is an  $\mathcal{F}(M)$ -linear operator on  $\Gamma(TM)$  and  $\nabla^t$  is a linear connection on the lightlike transverse vector bundle  $\text{tr}(TM)$ .

Let  $P$  denote the projection morphism of  $\Gamma(TM)$  on  $\Gamma(S(TM))$  with respect to the decomposition (2.2). We have

$$\nabla_X PY = \nabla_X^* PY + h^*(X, PY) \quad \forall X, Y \in \Gamma(TM), \tag{2.9}$$

$$\nabla_X U = -A_U^* X + \nabla_X^{*t} U \quad \forall X \in \Gamma(TM), \forall U \in \Gamma(TM^\perp), \tag{2.10}$$

where  $\nabla_X^*PY$  and  $A_U^*X$  belong to  $\Gamma(S(TM))$ ,  $\nabla^*$  and  $\nabla^{*t}$  are linear connections on  $S(TM)$  and  $TM^\perp$ , respectively,  $h^*$  is a  $\Gamma(TM^\perp)$ -valued  $\mathcal{F}(M)$ -bilinear form on  $\Gamma(TM) \times \Gamma(S(TM))$ , and  $A_U^*$  is a  $\Gamma(S(TM))$ -valued  $\mathcal{F}(U)$ -linear operator on  $\Gamma(TM)$ . They are the second fundamental form and the shape operator of the screen distribution  $S(TM)$ , respectively. Note that some authors called  $A_U^*$  (or  $A_N$ ) the Burali-Forti affinator [3, 9].

Define the following also on  $U$ :

$$C(X, PY) = \bar{g}(h^*(X, PY), N), \tag{2.11}$$

$$\varepsilon(X) = \bar{g}(\nabla_X^{*t}\xi, N). \tag{2.12}$$

One can verify that  $\varepsilon(X) = -\tau(X)$ . Thus, locally, one has

$$\nabla_X PY = \nabla_X^*PY + C(X, PY)\xi, \tag{2.13}$$

$$\nabla_X \xi = -A_\xi^*X - \tau(X)\xi, \quad \forall X \in \Gamma(TM). \tag{2.14}$$

**DEFINITION 2.2.** The screen distribution  $S(TM)$  is said to be parallel with respect to the induced connection  $\nabla$  if  $\nabla_X PY \in \Gamma(S(TM))$  for all  $X, Y \in \Gamma(TM)$ .

We will make use of the following theorem.

**THEOREM 2.3** (Duggal and Bejancu [11, page 89]). *Let  $(M, g, S(TM))$  be a lightlike hypersurface of  $(\tilde{M}, \tilde{g})$ . Then the following assertions are equivalent:*

- (i)  $S(TM)$  is parallel with respect to the induced connection  $\nabla$ ;
- (ii)  $h^*$  vanishes identically on  $M$ ;
- (iii)  $A_N$  vanishes identically on  $M$ .

Then it is easy to see that the following relations hold:

$$B(X, \xi) = 0 \quad \forall X \in \Gamma(TM|_U), \tag{2.15}$$

$$B(X, Y) = g(A_\xi^*X, Y), \tag{2.16}$$

$$A_\xi^*\xi = 0. \tag{2.17}$$

**3. Pseudoinverse of a degenerate metric.** We start this section by the construction of the isomorphisms  $\#$  and  $\flat$  with respect to the noninvertible induced metric  $g$ .

First, we consider on  $M$  the 1-form  $\eta$  defined by

$$\eta(\cdot) = \bar{g}(N, \cdot), \tag{3.1}$$

where  $N$  is given by (2.1). Let  $P$  be the projection morphism of  $\Gamma(TM)$  on  $\Gamma(S(TM))$  in (2.2). We have, for all  $X \in \Gamma(TM)$ ,

$$X = PX + \eta(X)\xi \tag{3.2}$$

and from (2.1) and (3.1),

$$\eta(X) = 0 \iff X \in \Gamma(STM). \tag{3.3}$$

Now, we define  $\flat$  by

$$\begin{aligned} \flat : \Gamma(TM) &\longrightarrow \Gamma(T^*M), \\ X &\longmapsto X^\flat \end{aligned} \tag{3.4}$$

such that

$$X^\flat = g(PX, \cdot) + \eta(X)\eta(\cdot). \tag{3.5}$$

It is obvious that  $\flat$  is linear.

As  $TM$  and  $T^*M$  have the same finite-constant rank, it suffices to show that  $\flat$  is injective, that is,  $X^\flat = 0 \iff X = 0$ , to get the bijection of  $\flat$ . Consider  $X \in \Gamma(TM)$  and assume  $X^\flat = 0$ . Then

$$X^\flat(Y) = 0 \quad \forall Y \in \Gamma(TM) \iff g(PX, Y) + \eta(X)\eta(Y) = 0, \quad \forall Y \in \Gamma(TM). \tag{3.6}$$

In particular, for any  $Y \in \Gamma(S(TM))$ , we have

$$g(PX, Y) = 0 \tag{3.7}$$

since  $\eta(Y) = 0$ .

As  $\Gamma(S(TM))$  is nondegenerate and  $PX \in \Gamma(STM)$ , we have

$$PX = 0. \tag{3.8}$$

Therefore,  $X$  satisfies

$$\eta(X)\eta(Y) = 0 \quad \forall Y \in \Gamma(TM). \tag{3.9}$$

In particular, if  $Y = \xi$ ,  $\eta(X) = 0$ . Hence,  $X = PX + \eta(X)\xi = 0$  and  $\flat$  is injective. We conclude that the so-defined  $\flat$  is an isomorphism of  $\Gamma(TM)$  on  $\Gamma(T^*M)$ . We denote by  $\sharp$  its inverse isomorphism.

Note that the above construction generalizes the one of nondegenerate case for, in the latter case,  $S(TM)$  coincides with  $TM$ . As a consequence, the 1-form  $\eta$  vanishes identically and  $P$  becomes the identity map on  $TM$ .

We now define the *associate metric*  $\tilde{g}$  of  $g$ . Put

$$\tilde{g}(X, Y) = X^\flat(Y). \tag{3.10}$$

It is obvious that  $\tilde{g}$  is bilinear in  $X$  and  $Y$ . Furthermore, from (3.5), we have that  $X^\flat(Y) = Y^\flat(X)$ , then  $\tilde{g}$  is symmetric. Its inverse  $\tilde{g}^{-1}$ , which we denote by  $g^{[\cdot, \cdot]}$ , will be called *the pseudoinverse* of the degenerate metric  $g$  on  $M$ .

Note that in the case where  $g$  is nondegenerate, the associate metric  $\tilde{g}$  coincides with  $g$  and the pseudoinverse  $g^{[\cdot\cdot]}$  coincides with the dual  $g^*$  (or  $g^{-1}$ ) of  $g$ .

In what follows, we use the following range of indices:

$$i, j = 1, \dots, n, \quad \alpha, \beta, \gamma = 0, 1, \dots, n, \quad A, B, C, \dots = 0, 1, \dots, n + 1. \tag{3.11}$$

Now, consider a local coordinate system  $(x^0, \dots, x^{n+1})$  adapted to decompositions (2.2) and (2.3), that is,

$$\left( X_i := \frac{\partial}{\partial x^i}, \quad \xi := \frac{\partial}{\partial x^0}, \quad N := \frac{\partial}{\partial x^{n+1}} \right) \tag{3.12}$$

is a local quasiorthogonal basis of  $T\overline{M}|_M$  such that

$$\begin{aligned} TM^\perp &= \text{Span}\{\xi\}, & S(TM) &= \text{Span}\{X_1, \dots, X_n\}, \\ \text{tr}(TM) &= \text{Span}\{N\}. \end{aligned} \tag{3.13}$$

Using (3.1), (3.2), (3.3), and (3.5), we have a standard computation that gives

$$\begin{aligned} \left( \frac{\partial}{\partial x^0} \right)^b \left( \frac{\partial}{\partial x^0} \right) &= 0 + \eta \left( \frac{\partial}{\partial x^0} \right) \eta \left( \frac{\partial}{\partial x^0} \right) = 1 = dx^0 \left( \frac{\partial}{\partial x^0} \right), \\ \left( \frac{\partial}{\partial x^0} \right)^b (X_i) &= g(P\xi, X_i) + \eta(\xi)\eta(X_i) = 0 = dx^0 \left( \frac{\partial}{\partial x^i} \right), \end{aligned} \tag{3.14}$$

so

$$\left( \frac{\partial}{\partial x^0} \right)^b = dx^0, \tag{3.15}$$

and the first terms in (3.14) are  $\tilde{g}_{00}$  and  $\tilde{g}_{0i}$ , respectively. Hence we have

$$\tilde{g}_{00} = 1, \quad \tilde{g}_{0i} = \tilde{g}_{i0} = 0, \quad i = 1, \dots, n. \tag{3.16}$$

We also have

$$\left( \frac{\partial}{\partial x^i} \right)^b \left( \frac{\partial}{\partial x^k} \right) = g(PX_i, X_k) + \eta(X_i)\eta(X_k) = g_{ik}. \tag{3.17}$$

Consequently,

$$\tilde{g}_{ij} = g_{ij}, \quad i, j = 1, \dots, n. \tag{3.18}$$

Thus, with respect to the quasiorthonormal basis  $(\xi, X_1, \dots, X_n, N)$ , the matrices of  $\tilde{g}$  and  $g^{[\cdot, \cdot]}$  are given by

$$\begin{aligned} \tilde{g} &= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & g_{ij} & \\ 0 & & & \end{pmatrix}, \\ g^{[\cdot, \cdot]} &= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & (g_{ij})^{-1} & \\ 0 & & & \end{pmatrix}. \end{aligned} \tag{3.19}$$

We have

$$g^{[\cdot, \cdot]} \cdot g = g \cdot g^{[\cdot, \cdot]} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & 1 \end{pmatrix}, \tag{3.20}$$

and this justifies the terminology pseudoinverse.

By (3.17) and (3.18), we have

$$\left(\frac{\partial}{\partial x^i}\right)^b \left(\frac{\partial}{\partial x^k}\right) = g_{ik} = \tilde{g}_{ij} dx^j \left(\frac{\partial}{\partial x^k}\right), \quad i, j = 1, \dots, n. \tag{3.21}$$

Hence

$$\left(\frac{\partial}{\partial x^i}\right)^b = \tilde{g}_{ij} dx^j. \tag{3.22}$$

Taking into account (3.15) and (3.22), we have

$$dx^\beta = g^{[\alpha\beta]} \left(\frac{\partial}{\partial x^\alpha}\right)^b, \quad \alpha, \beta = 0, \dots, n. \tag{3.23}$$

From (3.19) and (3.23), remark that if  $\rho$  is an endomorphism of  $TM$  (resp., a bilinear form on  $TM$ ), its trace with respect to  $g$  is given by

$$\begin{aligned} \text{trace}_g \rho &= \sum_{\alpha=0}^n \tilde{g}(\rho(X_\alpha), X_\alpha), \\ \text{trace}_g \rho &= g^{[\alpha\beta]} \rho_{\alpha\beta}, \end{aligned} \tag{3.24}$$

respectively.

We use (3.19) and (3.23) to deduce that

$$\begin{aligned} \tilde{g}(\xi, \xi) &= 1, \\ \tilde{g}(X, Y) &= g(X, Y) \quad \forall X, Y \in S(TM), \\ \tilde{g}(\xi, X) &= \eta(X) \quad \forall X \in TM. \end{aligned} \tag{3.25}$$

Now, let  $f : M \rightarrow \mathbb{R}$  be a smooth function on  $M$ . One defines intrinsically the gradient of  $f$  by  $\text{grad}^g f = (df)^\#$ . But

$$df = \frac{\partial f}{\partial x^\alpha} dx^\alpha \implies (df)^\# = \frac{\partial f}{\partial x^\alpha} (dx^\alpha)^\#. \tag{3.26}$$

Hence, from (3.23), we infer that

$$\begin{aligned} \text{grad}^g f &= \frac{\partial f}{\partial x^\alpha} g^{[\alpha\beta]} \left( \left( \frac{\partial}{\partial x^\beta} \right)^\flat \right)^\# = \frac{\partial f}{\partial x^\alpha} g^{[\alpha\beta]} \frac{\partial}{\partial x^\beta}, \\ \text{grad}^g f &= g^{[\alpha\beta]} \frac{\partial f}{\partial x^\alpha} \frac{\partial}{\partial x^\beta}. \end{aligned} \tag{3.27}$$

Let  $X$  be a smooth vector field defined on  $\mathcal{U} \subset M$ . The divergence  $\text{div}^g X$  of  $X$  with respect to the degenerate metric  $g$  is intrinsically defined by

$$\text{div}^g X = - \sum_{\alpha=0}^n \varepsilon_\alpha X_\alpha^\flat (\nabla_{X_\alpha} X). \tag{3.28}$$

Therefore, from (3.10), we have

$$\text{div}^g X = - \sum_{\alpha=0}^n \varepsilon_\alpha \tilde{g}(\nabla_{X_\alpha} X, X_\alpha) \tag{3.29}$$

with  $\varepsilon_0 = 1$ .

In summary, we have proved the following proposition.

**PROPOSITION 3.1.** *Let  $(M, g, S(TM))$  be a lightlike hypersurface of a semi-Riemannian  $(n + 2)$ -dimensional manifold  $(\bar{M}, \tilde{g})$ . There exists an associate metric  $\tilde{g}$  and a pseudoinverse  $g^{[\cdot]}$  of  $g$  on  $M$  such that locally on  $\mathcal{U} \subset M$ , the following holds:*

(i) *for any smooth function  $f : \mathcal{U} \subset M \rightarrow \mathbb{R}$ ,*

$$\text{grad}^g f = g^{[\alpha\beta]} f_\alpha \partial_\beta, \tag{3.30}$$

*where  $f_\alpha = \partial f / \partial x^\alpha$ ,  $\partial_\beta = \partial / \partial x^\beta$ ,  $\alpha, \beta = 0, \dots, n$ ;*

(ii) *for any vector field  $X$  on  $\mathcal{U} \subset M$ ,*

$$\text{div}^g X = - \sum_{\alpha=0}^n \varepsilon_\alpha \tilde{g}(\nabla_{X_\alpha} X, X_\alpha), \quad \varepsilon_0 = 1; \tag{3.31}$$

(iii) for a smooth function  $f$  defined on  $\mathcal{U} \subset M$ ,

$$\Delta^g f = - \sum_{\alpha=0}^n \varepsilon_\alpha \tilde{g}(\nabla_{X_\alpha} \text{grad}^g f, X_\alpha). \tag{3.32}$$

Here  $\Delta^g$  is the d'Alembertian with respect to  $g$  on  $\mathcal{U}$ .

Note that if the section  $\xi$  is globally defined on  $M$ , then our result is also global.

**4. Compatibility of  $\tilde{g}$  with respect to the connection  $\nabla$  induced by  $g$ .** It is well known that the induced metric  $g$  is not compatible with the induced connection  $\nabla$  in general, and this compatibility arises if and only if the lightlike hypersurface  $M$  is totally geodesic in  $\bar{M}$ . This section deals with the compatibility of the associate metric  $\tilde{g}$  with respect to  $\nabla$ .

For  $X, Y, Z \in \Gamma(TM)$ , we have

$$\begin{aligned} &(\nabla_X \tilde{g})(Y, Z) \\ &= X \cdot \tilde{g}(Y, Z) - \tilde{g}(\nabla_X Y, Z) - \tilde{g}(Y, \nabla_X Z) \\ &= X \cdot \{ \tilde{g}(PY, PZ) + \tilde{g}(PY, \eta(Z)\xi) + \tilde{g}(\eta(Y)\xi, PZ) + \eta(Y)\eta(Z)\tilde{g}(\xi, \xi) \} \\ &\quad - \tilde{g}(\nabla_X(PY + \eta(Y)\xi), PZ + \eta(Z)\xi) - \tilde{g}(PY + \eta(Y)\xi, \nabla_X(PZ + \eta(Z)\xi)) \\ &= X \cdot \{ g(PY, PZ) + \eta(Z)\eta(PY) + \eta(Y)\eta(PZ) + \eta(Y)\eta(Z) \} \\ &\quad - \tilde{g}(\nabla_X PY + \nabla_X(\eta(Y)\xi), PZ) - \tilde{g}(\nabla_X PY + \nabla_X(\eta(Y)\xi), \eta(Z)\xi) \\ &\quad - \tilde{g}(PY, \nabla_X PZ + \nabla_X(\eta(Z)\xi)) - \tilde{g}(\eta(Y)\xi, \nabla_X PZ + \nabla_X(\eta(Z)\xi)) \\ &= X \cdot \{ g(PY, PZ) + \eta(Y)\eta(Z) \} - \tilde{g}(\nabla_X PY, PZ) - \tilde{g}(\nabla_X(\eta(Y)\xi), PZ) \\ &\quad - \tilde{g}(\nabla_X PY, \eta(Z)\xi) - \tilde{g}(\nabla_X(\eta(Y)\xi), \eta(Z)\xi) - \tilde{g}(PY, \nabla_X PZ) \\ &\quad - \tilde{g}(PY, \nabla_X(\eta(Z)\xi)) - \tilde{g}(\eta(Y)\xi, \nabla_X PZ) - \tilde{g}(\eta(Y)\xi, \nabla_X(\eta(Z)\xi)). \end{aligned} \tag{4.1}$$

Now use (2.9), (2.11), and (3.10) to obtain

$$\begin{aligned} (\nabla_X \tilde{g})(Y, Z) &= X \cdot \{ g(PY, PZ) \} + \eta(Y)X \cdot \eta(Z) + \eta(Z)X \cdot \eta(Y) \\ &\quad - \tilde{g}(\overset{*}{\nabla}_X PY, PZ) - \tilde{g}(C(X, PY)\xi, PZ) \\ &\quad - g(\overset{*}{\nabla}_X PY, PZ) - \eta(PZ)C(X, PY) - (X \cdot \eta(Y))\eta(PZ) \\ &\quad - \eta(Y)\tilde{g}(-A_\xi^*, PZ) - \eta(Y)\tilde{g}(-\tau(X)\xi, PZ) \\ &\quad - \eta(Z)\eta(\overset{*}{\nabla}_X PY) - \eta(Z)C(X, PY) - \eta(Z)X \cdot \eta(Y) \end{aligned}$$

$$\begin{aligned}
 & -\eta(Y)\eta(Z)\tilde{g}\left(-\overset{*}{A}_\xi X, \xi\right) - \eta(Y)\eta(Z)\tilde{g}\left(-\tau(X)\xi, \xi\right) \\
 & -g\left(PY, \overset{*}{\nabla}_X PZ\right) - C(X, PZ)\eta(PY) - \eta(PY)X \cdot \eta(Z) \\
 & -\eta(Z)\tilde{g}\left(PY, \overset{*}{A}_\xi X\right) - \eta(Z)\tilde{g}\left(-PY, -\tau(X)\xi\right) \\
 & -\eta(Y)\eta\left(\overset{*}{\nabla}_X PZ\right) - \eta(Y)C(X, PZ) - \eta(Y)X \cdot (\eta(Z)) \\
 & -\eta(Z)\eta(Y)\tilde{g}\left(\xi, -\overset{*}{A}_\xi X\right) - \eta(Z)\eta(Y)\tilde{g}\left(\xi, -\tau(X)\xi\right).
 \end{aligned} \tag{4.2}$$

But from (3.3), we know that  $\eta(X) = 0$  if and only if  $X \in \Gamma(S(TM))$ . Using (2.16) and (3.2) leads to

$$\begin{aligned}
 (\nabla_X \tilde{g})(Y, Z) &= X \cdot \{g(PY, PZ)\} + \eta(Y)X \cdot \eta(Z) + \eta(Z)X \cdot \eta(Y) \\
 & - \tilde{g}\left(\overset{*}{\nabla}_X PY, PZ\right) + \eta(Y)B(X, PZ) - \eta(Z)C(X, PY) \\
 & - \eta(Z)X \cdot (\eta(Y)) + \tau(X)\eta(Y)\eta(Z) - g\left(PY, \overset{*}{\nabla}_X PZ\right) \\
 & + \eta(Z)B(X, PY) - \eta(Y)C(X, PZ) - \eta(Y)X \cdot (\eta(Z)) \\
 & + \tau(X)\eta(Y)\eta(Z).
 \end{aligned} \tag{4.3}$$

The above expression reduces to

$$\begin{aligned}
 (\nabla_X \tilde{g})(Y, Z) &= \left\{X \cdot g(PY, PZ) - g\left(\overset{*}{\nabla}_X PY, PZ\right) - g\left(PY, \overset{*}{\nabla}_X PZ\right)\right\} \\
 & + \eta(Y)B(X, PZ) + \eta(Z)B(X, PY) - \eta(Z)C(X, PY) \\
 & - \eta(Y)C(X, PZ) + 2\tau(X)\eta(Y)\eta(Z).
 \end{aligned} \tag{4.4}$$

Now, recall that since the connection  $\overset{*}{\nabla}$  is compatible with  $g$ , the expression between brackets is zero, and therefore we have

$$\begin{aligned}
 (\nabla_X \tilde{g})(Y, Z) &= \eta(Y)(B(X, PZ) - C(X, PZ)) \\
 & + \eta(Z)(B(X, PY) - C(X, PY)) \\
 & + 2\tau(X)\eta(Y)\eta(Z) \quad \forall X, Y, Z \in \Gamma(TM).
 \end{aligned} \tag{4.5}$$

Then we get the following theorem.

**THEOREM 4.1.** *Let  $(M, g, S(TM))$  be a lightlike hypersurface of a semi-Riemannian manifold  $(\bar{M}, \tilde{g})$ . The induced connection  $\nabla$  by  $g$  on  $M$  is compatible with the associate metric  $\tilde{g}$  of  $g$  if and only if, for all  $X, Z \in \Gamma(TM)$ ,*

$$B(X, PZ) = C(X, PZ), \quad \tau(X) = 0. \tag{4.6}$$

**PROOF.** It is well known that  $\nabla$  is compatible with  $\tilde{g}$  if and only if  $(\nabla_X \tilde{g})(Y, Z) = 0$  for all  $X, Y, Z \in \Gamma(TM)$ . Putting  $Y = Z = \xi$  in (4.5) gives  $\tau(X) = 0$  for all  $X \in \Gamma(TM)$ , and setting  $Y = \xi$  yields  $B(X, PZ) = C(X, PZ)$  for all  $X, Z \in \Gamma(TM)$ .  $\square$

We derive from (4.5) the main covariant derivative formula

$$X \cdot \tilde{g}(X, Z) = \tilde{g}(\nabla_X Y, Z) + \tilde{g}(Y, \nabla_X Z) + \eta(Y)(B(X, PZ) - C(X, PZ)) + \eta(Z)(B(X, PY) - C(X, PY)) + 2\tau(X)\eta(Y)\eta(Z). \tag{4.7}$$

We discuss here the effect of the change of the screen distribution on the compatibility of  $\tilde{g}$  with  $g$ . The first equation in (4.6) indicates that only some privileged changes in the screen distribution maintain the compatibility property of the associate metric  $\tilde{g}$  with  $g$ . For two screen distributions  $S(TM)$  and  $S(TM)'$  on  $(M, g)$ , with two orthogonal frame fields  $(W_i)_{i=1, \dots, n}$  and  $(W'_i)_{i=1, \dots, n}$ , respectively, the relationship between the second fundamental forms  $C$  and  $C'$  of the screen distributions  $S(TM)$  and  $S(TM)'$ , respectively, is given by (see [4])

$$C'(X, PY) = C(X, PY) - \frac{1}{2} \langle W, W \rangle B(X, Y) + g(\nabla_X PY, W), \tag{4.8}$$

where  $W = \sum_{i=1}^n c_i W_i$  is the characteristic vector field of the screen change.

Therefore, a change in screen distribution leaves the compatibility of  $\tilde{g}$  with  $g$  invariant if and only if

$$\omega \left( \nabla_X PY - \frac{1}{2} B(X, Y) W \right) = 0 \quad \forall X, Y \in \Gamma(TM), \tag{4.9}$$

where  $\omega$  is the dual 1-form of  $W$  defined by  $\omega(\cdot) = g(W, \cdot)$ .

From (3.1) and (3.10), if  $\tilde{g}$  is associated to  $g$  on  $S(TM)$  and  $\tilde{g}'$  is associated to  $g$  on  $S(TM)'$ , we have

$$\tilde{g}'(X, Y) = \tilde{g}(X, Y) + D(X, Y), \tag{4.10}$$

where  $D$  is the bilinear form defined by

$$D(X, Y) = \eta(X)\omega(Y) + \eta(Y)\omega(X) + \omega(X)\omega(Y). \tag{4.11}$$

Before starting the next section, it is important to stress that the 1-form  $\tau$  of (4.6), defined in (2.6) and (2.8), depends on the normalization of the rank-one subbundle  $\text{Rad}(TM)$ , that is, on the choice of  $\xi \in \Gamma(TM^\perp|_{\mathcal{U}})$ . It has been proved in [11, page 99] that there exists a pair (a normalization)  $\{\xi, N\}$  on  $\mathcal{U}$  such that the corresponding 1-form  $\tau$  vanishes identically.

From now on we use this normalization. Then, relations (2.6) and (2.14) become

$$\tilde{\nabla}_X N = -A_N X, \tag{4.12}$$

$$\nabla_X \xi = -\overset{*}{A}_\xi X, \tag{4.13}$$

for all  $X \in \Gamma(TM)$

**5. Laplacian of lightlike hypersurfaces endowed with parallel screen distribution.** Let  $(M^{n+1}, g, S(TM))$  be a lightlike hypersurface of the Lorentzian space  $\mathbb{R}_1^{n+2}$ . The metric on  $\mathbb{R}_1^{n+2}$  and the position vector field of  $M^{n+1}$  will be denoted by  $\langle \cdot, \cdot \rangle$  and  $x$ , respectively.

We have

$$\begin{aligned} TM &= \text{Rad}(TM) \perp S(TM) = TM^\perp \perp S(TM), \\ T\mathbb{R}_1^{n+2}|_M &= (\text{Rad}(TM) \oplus \text{tr}(TM)) \perp S(TM), \end{aligned} \tag{5.1}$$

and  $(X_0 := \xi, N, X_1, \dots, X_n)$  is a local quasiorthonormal frame field on  $\mathbb{R}_1^{n+2}$  such that at each point  $x \in M$ ,  $(X_1, \dots, X_n)$  represents a basis of  $S(T_x M)$ . As the plane spanned by the pair  $\{\xi_x, N_x\}$  is a hyperbolic plane at any  $x \in M$  and  $\mathbb{R}_1^{n+2}$  has index  $q = 1$ , it follows that the screen distribution on  $M \subset \mathbb{R}_1^{n+2}$  is Riemannian, that is, the induced metric on  $S(TM)$  is positive definite. As a consequence, the signature  $\varepsilon_k$  of  $X_k$ ,  $k = 1, \dots, n$ , is  $\varepsilon_k = 1$ .

Define on  $M$  the smooth functions

$$f_0(x) = \langle x, N \rangle, \quad f_{n+1}(x) = \langle x, \xi \rangle, \quad f_k(x) = \langle x, X_k \rangle, \quad k = 1, \dots, n, \tag{5.2}$$

where  $x$  stands for the position vector field of  $M$  in  $\mathbb{R}_1^{n+2}$ . We have

$$U_A := \text{grad}^g f_A = g^{[\alpha\gamma]} f_{A;\alpha} X_\gamma, \tag{5.3}$$

where  $f_{A;\alpha} := X_\alpha \cdot f_A$ .

The position vector field  $x$  can be written in  $\mathbb{R}_1^{n+2}$  as

$$x = (\langle x, N \rangle, \langle x, \xi \rangle, \varepsilon_1 \langle x, X_1 \rangle, \dots, \varepsilon_n \langle x, X_n \rangle) \tag{5.4}$$

with respect to  $(\xi, N, X_1, \dots, X_n)$ . As  $\varepsilon_k = 1$ ,  $k = 1, \dots, n$ , we have

$$x = (f_0(x), f_{n+1}(x), f_1(x), \dots, f_n(x)). \tag{5.5}$$

Let  $\Delta^g$  denote the Laplace operator on functions with respect to the degenerate metric  $g$ . Then

$$\Delta^g x = (\Delta^g f_0(x), \Delta^g f_N(x), \Delta^g f_1(x), \dots, \Delta^g f_{n+1}(x)). \tag{5.6}$$

But

$$\Delta^\theta f_A = - \sum_{\alpha=0}^n \varepsilon_\alpha \tilde{g}(\nabla_{X_\alpha} U_A, X_\alpha). \tag{5.7}$$

According to the above remark on the signature and (2.1), we have here  $\varepsilon_\alpha = 1$  for all  $\alpha$ .

Using (4.7) yields

$$\begin{aligned} \tilde{g}(\nabla_{X_\alpha} U_A, X_\alpha) &= X_\alpha \cdot \tilde{g}(U_A, X_\alpha) - \tilde{g}(U_A, \nabla_{X_\alpha} X_\alpha) \\ &\quad - \eta(U_A)(B(X_\alpha, PX_\alpha) - C(X_\alpha, PX_\alpha)) \\ &\quad - \eta(X_\alpha)(B(X_\alpha, PU_A) - C(X_\alpha, PU_A)) \\ &\quad - 2\tau(X_\alpha)\eta(X_\alpha)\eta(U_A). \end{aligned} \tag{5.8}$$

In this section, we assume  $S(TM)$  parallel. From Theorem 2.3 and (2.11), we have  $C = 0$  and  $A_N = 0$  on  $M$ . Moreover, we can write, without loss of generality,  $\nabla_X X_i = 0$  for all  $X \in \Gamma(TM)$ , where the  $X_i$ 's are vectors of a local basis of  $\Gamma(S(TM))$ . Finally, as  $\tau = 0$ , we have

$$\begin{aligned} \tilde{g}(\nabla_{X_\alpha} U_A, X_\alpha) &= X_\alpha \cdot \tilde{g}(U_A, X_\alpha) - \tilde{g}(U_A, \nabla_{X_\alpha} X_\alpha) \\ &\quad - \eta(U_A)B(X_\alpha, PX_\alpha) - \eta(X_\alpha)B(X_\alpha, PU_A), \quad \alpha = 0, \dots, n. \end{aligned} \tag{5.9}$$

But

$$\begin{aligned} X_\alpha \cdot \tilde{g}(U_A, X_\alpha) &= X_\alpha \cdot \{ \tilde{g}(g^{[\varepsilon\gamma]} f_{A;\varepsilon} X_\gamma, X_\alpha) \} \\ &= X_\alpha \cdot \{ f_{A;\varepsilon} g^{[\varepsilon\gamma]} \tilde{g}_{\gamma\alpha} \} = X_\alpha \cdot \{ \delta_{\alpha\varepsilon}^\varepsilon f_{A;\varepsilon} \}. \end{aligned} \tag{5.10}$$

Hence

$$X_\alpha \cdot \tilde{g}(U_A, X_\alpha) = X_\alpha \cdot \{ f_{A;\alpha} \} =: f_{A;\alpha\alpha}. \tag{5.11}$$

Now we compute  $f_{A;\alpha}$  and  $f_{A;\alpha\alpha}$  for  $A = 0, \dots, n+1$  and  $\alpha = 0, \dots, n$ :

$$f_0(x) = \langle x, N \rangle \implies f_{0;\alpha} = X_\alpha \cdot \langle x, N \rangle = \langle \tilde{\nabla}_{X_\alpha} x, N \rangle + \langle x, \tilde{\nabla}_{X_\alpha} N \rangle. \tag{5.12}$$

Note that as  $\tau = 0$  and  $S(TM)$  is parallel,  $N$  is a parallel vector field. Indeed,  $\tilde{\nabla}_X N = -A_N X + \tau(X)N$ . It follows from (4.12) and Theorem 2.3 that

$$\tilde{\nabla}_X N = 0 \quad \forall X \in \Gamma(TM). \tag{5.13}$$

Hence

$$\begin{aligned} f_{0;\alpha} &= \langle X_\alpha, N \rangle, \\ f_{0;\alpha\alpha} &= X_\alpha \cdot \langle X_\alpha, N \rangle \\ &= \langle \tilde{\nabla}_{X_\alpha} X_\alpha, N \rangle + \langle X_\alpha, \tilde{\nabla}_{X_\alpha} N \rangle \\ &= \langle \nabla_{X_\alpha} X_\alpha + B(X_\alpha, X_\alpha)N, N \rangle. \end{aligned} \tag{5.14}$$

Thus

$$\begin{aligned}
 f_{0;\alpha\alpha} &= \langle \nabla_{X_\alpha} X_\alpha, N \rangle, \\
 f_{n+1}(x) = \langle x, \xi \rangle &\implies f_{n+1;\alpha} = \langle X_\alpha, \xi \rangle + \langle x, \bar{\nabla}_{X_\alpha} \xi \rangle.
 \end{aligned}
 \tag{5.15}$$

Therefore,

$$\begin{aligned}
 f_{n+1;\alpha} &= \langle x, \bar{\nabla}_{X_\alpha} \xi \rangle, \\
 f_{n+1;\alpha\alpha} &= X_\alpha \cdot \langle x, \bar{\nabla}_{X_\alpha} \xi \rangle \\
 &= \langle X_\alpha, \bar{\nabla}_{X_\alpha} \xi \rangle + \langle x, \bar{\nabla}_{X_\alpha} \bar{\nabla}_{X_\alpha} \xi \rangle \\
 &= -B(X_\alpha, X_\alpha) + \langle x, \bar{\nabla}_{X_\alpha} \bar{\nabla}_{X_\alpha} \xi \rangle.
 \end{aligned}
 \tag{5.16}$$

But a straightforward computation leads to

$$\bar{\nabla}_{X_\alpha} \bar{\nabla}_{X_\alpha} \xi = -\nabla_{X_\alpha}^* \bar{A}_\xi X_\alpha + B(\nabla_{X_\alpha} \xi, X_\alpha) N.
 \tag{5.17}$$

We also obtain

$$\begin{aligned}
 \langle -\nabla_{X_\alpha}^* \bar{A}_\xi X_\alpha, X_i \rangle &= -X_\alpha \cdot \langle \bar{A}_\xi X_\alpha, X_i \rangle + B(X_\alpha, \bar{\nabla}_{X_\alpha} X_i) \\
 &= -X_\alpha \cdot B(X_\alpha, X_i),
 \end{aligned}
 \tag{5.18}$$

where we have used

$$\bar{\nabla}_{X_\alpha} X_i = \nabla_{X_\alpha} X_i - C(X_\alpha, X_i) = \nabla_{X_\alpha} X_i = 0.
 \tag{5.19}$$

Therefore,

$$\begin{aligned}
 \langle x, \bar{\nabla}_{X_\alpha} \bar{\nabla}_{X_\alpha} \xi \rangle &= B(\nabla_{X_\alpha} \xi, X_\alpha) \langle x, N \rangle - \sum_{i=1}^n \langle x, X_i \rangle X_\alpha \cdot B(X_\alpha, X_i) \\
 &= -B(\bar{A}_\xi X_\alpha, X_\alpha) \langle x, N \rangle - \sum_{i=1}^n \langle x, X_i \rangle X_\alpha \cdot B(X_\alpha, X_i),
 \end{aligned}
 \tag{5.20}$$

so

$$\begin{aligned}
 f_{n+1;\alpha\alpha} &= -B(X_\alpha, X_\alpha) - B(\bar{A}_\xi X_\alpha, X_\alpha) \langle x, N \rangle \\
 &\quad - \sum_{i=1}^n \langle x, X_i \rangle X_\alpha \cdot B(X_\alpha, X_i), \\
 f_k(x) = \langle x, X_k \rangle &\implies f_{k;\alpha} = \langle X_\alpha, X_k \rangle + \langle x, \bar{\nabla}_{X_\alpha} X_k \rangle, \\
 f_{k;\alpha} &= \langle X_\alpha, X_k \rangle + \langle x, B(X_\alpha, X_k) N \rangle, \\
 f_{k;\alpha\alpha} &= \langle \bar{\nabla}_{X_\alpha} X_\alpha, X_k \rangle + \langle X_\alpha, \bar{\nabla}_{X_\alpha} X_k \rangle \\
 &\quad + X_\alpha \cdot B(X_\alpha, X_k) \langle x, N \rangle + B(X_\alpha, X_k) \langle X_\alpha, N \rangle.
 \end{aligned}
 \tag{5.21}$$

Moreover,

$$\langle \bar{\nabla}_{X_\alpha} X_\alpha, X_k \rangle = \langle \nabla_{X_\alpha} X_\alpha + B(X_\alpha, X_\alpha)N, X_k \rangle = \langle \nabla_{X_\alpha} X_\alpha, X_k \rangle. \tag{5.22}$$

But  $\nabla_{X_\alpha} X_\alpha = 0$  for all  $\alpha$ . Indeed, if  $\alpha = 0$ , from (2.14) and (2.17),  $\nabla_\xi \xi = 0$ . For  $\alpha = i \neq 0$ ,  $\nabla_{X_i} X_i = 0$  from the condition of parallel screen distribution. So  $\langle \bar{\nabla}_{X_\alpha} X_\alpha, X_k \rangle = 0$ . We also have that

$$\langle X_\alpha, \bar{\nabla}_{X_\alpha} X_k \rangle = 0, \quad B(X_\alpha, X_k) \langle X_\alpha, N \rangle = 0 \tag{5.23}$$

for at least one of  $B(X_\alpha, X_k)$  or  $\langle X_\alpha, N \rangle$  being zero according to either  $\alpha = 0$  or  $\alpha \neq 0$ , so

$$f_{k;\alpha\alpha} = [X_\alpha \cdot B(X_\alpha, X_k)] \langle x, N \rangle. \tag{5.24}$$

From  $\nabla_{X_\alpha} X_\alpha = 0$ , the term  $\tilde{g}(U_A, \nabla_{X_\alpha} X_\alpha)$  in expression (5.9) is zero.

Next we compute the term  $\tilde{g}(\nabla_{X_\alpha} U_A, X_\alpha)$  in the left-hand side of (5.9) for  $A = 0, \dots, n + 1$  and  $\alpha = 0, \dots, n$ :

$$\tilde{g}(\nabla_\xi U_0, \xi) = f_{0;00} - \tilde{g}(U_0, \nabla_\xi \xi) - \eta(U_0 B(\xi, P\xi)) - \eta(\xi) B(\xi, P U_0) = 0. \tag{5.25}$$

Hence

$$\tilde{g}(\nabla_\xi U_0, \xi) = 0. \tag{5.26}$$

Similar computations give the other terms as follows:

$$\begin{aligned} \tilde{g}(\nabla_{X_k} U_0, X_k) &= -B(X_k, X_k), & \tilde{g}(\nabla_\xi U_k, \xi) &= 0, \\ \tilde{g}(\nabla_{X_l} U_k, X_l) &= [X_l \cdot B(X_l, X_k)] \langle x, N \rangle, & \tilde{g}(\nabla_\xi U_{n+1}, \xi) &= 0, \\ \tilde{g}(\nabla_{X_l} U_{n+1}, X_l) &= -B(X_l, X_l) - B(A_\xi^* X_l, X_l) \langle x, N \rangle \\ &\quad - \sum_{i=1}^n \langle x, X_i \rangle X_l \cdot B(X_l, X_i). \end{aligned} \tag{5.27}$$

We will make use of the following lemma.

**LEMMA 5.1.** *Let  $(M, g, S(TM))$  be a lightlike hypersurface of  $\mathbb{R}_1^{n+2}$ . Set  $B_{jk;i} := X_i \cdot B(X_j, X_k)$ . If  $S(TM)$  is parallel and  $\tau = 0$ , then  $B_{jk;i}$  is symmetric with respect to the indices  $i, j$ , and  $k$ .*

**PROOF.** Clearly,  $B_{jk;i}$  is symmetric with respect to  $j$  and  $k$ . It suffices to show its symmetry with respect to  $i$  and  $j$ . First,

$$(\nabla_{X_i} B)(X_j, X_k) = X_i \cdot B(X_j, X_k) - B(\nabla_{X_i} X_j, X_k) - B(X_j, \nabla_{X_i} X_k). \tag{5.28}$$

Since  $S(TM)$  is parallel, the two last terms of this equality are zero. Hence, for all  $i, j, k = 1, \dots, n$ ,

$$(\nabla_{X_i} B)(X_j, X_k) = X_i \cdot B(X_j, X_k). \tag{5.29}$$

Next, Gauss-Codazzi equation gives

$$\begin{aligned} \tilde{g}(\bar{R}(X_i, X_j)X_k, \xi) &= (\nabla_{X_i}B)(X_j, X_k) - (\nabla_{X_j}B)(X_i, X_k) \\ &\quad + B(X_j, X_k)\tau(X_i) - B(X_i, X_k)\tau(X_j). \end{aligned} \tag{5.30}$$

The ambient space is  $\mathbb{L}^{n+2} := \mathbb{R}_1^{n+2}$ , hence  $\bar{R} = 0$ . As  $\tau = 0$ , we get

$$(\nabla_{X_i}B)(X_j, X_k) = (\nabla_{X_j}B)(X_i, X_k), \tag{5.31}$$

that is,

$$X_i \cdot B(X_j, X_k) = X_j \cdot B(X_i, X_k), \tag{5.32}$$

which shows that  $B_{jk;i}$  is symmetric with respect to  $i$  and  $j$ . This achieves the proof.  $\square$

Now we are in a position to compute each term  $\Delta^g f_A$  in (5.7) for  $A = 0, \dots, n + 1$ . We have

$$\begin{aligned} \sum_{\alpha=0}^n \varepsilon_\alpha \tilde{g}(\nabla_{X_\alpha}U_0, X_\alpha) &= \tilde{g}(\nabla_\xi U_0, \xi) + \sum_{k=1}^n \varepsilon_k \tilde{g}(\nabla_{X_k}U_0, X_k) \\ &= - \sum_{k=1}^n B(X_k, X_k). \end{aligned} \tag{5.33}$$

Hence, using (2.15), we infer that

$$\sum_{\alpha=0}^n \varepsilon_\alpha \tilde{g}(\nabla_{X_\alpha}U_0, X_\alpha) = - \sum_{\alpha=0}^n B(X_\alpha, X_\alpha). \tag{5.34}$$

We also compute similar sums for  $U_{n+1}$  and  $U_k$ :

$$\begin{aligned} \sum_{\alpha=0}^n \varepsilon_\alpha \tilde{g}(\nabla_{X_\alpha}U_{n+1}, X_\alpha) &= \tilde{g}(\nabla_\xi U_{n+1}, \xi) + \sum_{k=1}^n \varepsilon_k \tilde{g}(\nabla_{X_k}U_{n+1}, X_k) \\ &= - \sum_{k=1}^n B(X_k, B(X_k)) - \left[ \sum_{k=1}^n B(A_\xi^* X_k, X_k) \right] \langle x, N \rangle \\ &\quad - \sum_{k=1}^n \sum_{i=1}^n \langle x, X_i \rangle X_k \cdot B(X_k, X_i). \end{aligned} \tag{5.35}$$

By using Lemma 5.1 and (2.15), we obtain

$$\begin{aligned} \sum_{\alpha=0}^n \varepsilon_\alpha \tilde{g}(\nabla_{X_\alpha}U_{n+1}, X_\alpha) &= - \sum_{k=1}^n B(X_k, X_k) - \left[ \sum_{k=1}^n B(A_\xi^* X_k, X_k) \right] \langle x, N \rangle \\ &\quad - \sum_{i=1}^n \left[ X_i \cdot \sum_{\alpha=0}^n B(X_\alpha, X_\alpha) \right] \langle x, X_i \rangle. \end{aligned} \tag{5.36}$$

Finally,

$$\begin{aligned} \sum_{\alpha=0}^n \varepsilon_\alpha \tilde{g}(\nabla_{X_\alpha} U_k, X_\alpha) &= \tilde{g}(\nabla_\xi U_k, \xi) + \sum_{l=1}^n \varepsilon_l \tilde{g}(\nabla_{X_l} U_l, X_k) \\ &= \sum_{l=1}^n X_l \cdot B(X_l, X_k) \langle x, N \rangle, \end{aligned} \tag{5.37}$$

and using Lemma 5.1 and (2.15), we also obtain

$$\sum_{\alpha=0}^n \varepsilon_\alpha \tilde{g}(\nabla_{X_\alpha} U_k, X_\alpha) = \left[ X_k \cdot \sum_{\alpha=0}^n B(X_\alpha, X_\alpha) \right] \langle x, N \rangle. \tag{5.38}$$

Put

$$H = \frac{1}{n+1} \left[ \sum_{\alpha=0}^n B(X_\alpha, X_\alpha) \right] (\xi + N) \tag{5.39}$$

to obtain

$$\begin{aligned} \Delta^g x &= (n+1)H - \sum_{k=1}^n \langle x, N \rangle X_k \cdot \langle (n+1)H, N \rangle X_k \\ &+ \left\{ \sum_{\alpha=0}^n B(A_\xi^* X_\alpha, X_\alpha) \langle x, N \rangle - \sum_{i=1}^n X_i \cdot \sum_{\alpha=0}^n B(X_\alpha, X_\alpha) \langle x, X_i \rangle \right\} N. \end{aligned} \tag{5.40}$$

If we now put

$$K = \left[ \sum_{\alpha=0}^n B(X_\alpha, X_\alpha) \right], \tag{5.41}$$

then

$$\begin{aligned} \Delta^g x &= (n+1)H - \sum_{k=1}^n (\langle x, N \rangle X_k \cdot K) X_k \\ &+ \left\{ \sum_{\alpha=0}^n B(A_\xi^* X_\alpha, X_\alpha) \langle x, N \rangle - \sum_{k=1}^n \langle x, X_k \rangle X_k \cdot K \right\} N. \end{aligned} \tag{5.42}$$

Thus, we have established the following theorem.

**THEOREM 5.2.** *Let  $(M, g, S(TM))$  be a lightlike hypersurface of the Lorentzian space  $\mathbb{L}^{n+2} := \mathbb{R}_1^{n+2}$ , endowed with a parallel screen distribution  $S(TM)$  and with symmetric Ricci tensor. Then its Laplacian with respect to the degenerate*

metric  $g$  is given by

$$\Delta^g x = (n + 1)H - \sum_{k=1}^n (\langle x, N \rangle X_k \cdot K) X_k + \left\{ \sum_{\alpha=0}^n B(A_{\xi}^* X_{\alpha}, X_{\alpha}) \langle x, N \rangle - \sum_{k=1}^n \langle x, X_k \rangle X_k \cdot K \right\} N. \tag{5.43}$$

**DEFINITION 5.3.** On the lightlike hypersurface  $(M, g, S(TM))$ , the smooth function  $\sigma$  defined by

$$\sigma = \frac{1}{\sqrt{2}(n + 1)} K, \quad K = \sum_{\alpha}^n B(X_{\alpha}, X_{\alpha}), \tag{5.44}$$

is called the mean curvature function of  $M$ .

**6. An application.** It is well known that minimal immersions of smooth manifolds  $M$  into Euclidean spheres are those immersions whose coordinate functions in the ambient Euclidian space are eigenfunctions of the Laplacian in the induced metric with eigenvalue  $\lambda = -\dim(M)$ . Then the following general problem comes out in a natural way: can we classify lightlike submanifolds by means of some Laplacian differential equation involving the isometric immersion?

Our purpose in this section is to study lightlike hypersurfaces satisfying the eigenvalue equation  $\Delta^g x = \lambda x$ , where  $\lambda$  is a constant and  $x$  the position vector field. Our result here stands as follows.

**THEOREM 6.1.** *Let  $(M, g, S(TM))$  be a lightlike hypersurface of the Lorentzian space  $\mathbb{L}^{n+2} := \mathbb{R}_1^{n+2}$ , endowed with a parallel screen distribution  $S(TM)$  and with symmetric Ricci tensor. If  $(M, g, S(TM))$  satisfies the eigenvalue equation  $\Delta^g x = \lambda x$ , then*

- (1)  $\lambda = 0$ ;
- (2) the mean curvature function  $\sigma$  vanishes identically on  $M$ ;
- (3) the shape operator  $A_{\xi}^*$  of the screen distribution is trace-free with respect to the second fundamental form  $B$  of  $M$ , that is,

$$\sum_{\alpha=0}^n B(A_{\xi}^* X_{\alpha}, X_{\alpha}) = 0. \tag{6.1}$$

**PROOF.** If  $M$  is an eigenhypersurface of  $\Delta^g$  with eigenvalue  $\lambda$ , then we have, for all  $x \in M$ ,

$$\lambda \langle x, N \rangle = K, \tag{6.2}$$

$$\lambda \langle x, \xi \rangle = K + \langle x, N \rangle \sum_{\alpha=0}^n B(A_{\xi}^* X_{\alpha}, X_{\alpha}) - \sum_{k=1}^n \langle x, X_k \rangle (X_k \cdot K), \tag{6.3}$$

$$\lambda \langle x, X_k \rangle = \langle x, N \rangle X_k \cdot K. \tag{6.4}$$

Equation (6.2) leads to

$$\xi \cdot K = \lambda \langle \xi, N \rangle = \lambda, \tag{6.5}$$

but we have

$$\begin{aligned} \xi \cdot K &= \xi \cdot \left[ \sum_{\alpha=0}^n B(X_\alpha, X_\alpha) \right] = \sum_{k=1}^n \xi \cdot B(X_k, X_k) \\ &= \sum_{k=1}^n \{ (\nabla_\xi B)(X_k, X_k) + 2B(\nabla_\xi X_k, X_k) \} \\ &= \sum_{k=1}^n \{ (\nabla_\xi B)(X_k, X_k) \}. \end{aligned} \tag{6.6}$$

From Gauss-Codazzi equation, we have

$$(\nabla_\xi B)(X_k, X_k) = (\nabla_{X_k} B)(\xi, X_k). \tag{6.7}$$

This leads to

$$(\nabla_\xi B)(X_k, X_k) = B(A_\xi^* X_k, X_k), \tag{6.8}$$

and using this and (2.15), we deduce that

$$\xi \cdot K = \lambda = \sum_{\alpha=0}^n B(A_\xi^* X_\alpha, X_\alpha). \tag{6.9}$$

Then, if  $\lambda = 0$ , the first claim in Theorem 6.1 is obvious. The second one comes from (6.2) and Definition 5.3 of  $\sigma$ . Finally, the third assertion follows from relation (6.9).

Now assume  $\lambda \neq 0$ . The differentiation of equation (6.2) with respect to  $X_k$ ,  $k = 1, \dots, n$ , leads to

$$X_k \cdot K = \lambda \langle X_k, N \rangle = 0, \tag{6.10}$$

where we use  $\bar{\nabla}_X N = 0$  for all  $X \in \Gamma(TM)$  and (2.1). Hence we get

$$X_k \cdot K = 0, \quad k = 1, \dots, n. \tag{6.11}$$

Substitute (6.11) in (6.4) and  $\lambda \neq 0$  to get

$$\langle x, X_k \rangle = 0, \quad k = 1, \dots, n. \tag{6.12}$$

Hence, there exist two smooth functions  $r$  and  $l$  such that the position vector field along  $M$  is given by

$$x = r(x)\xi + l(x)N. \tag{6.13}$$

Using (6.11), (6.9), and (6.2) in (6.3) leads to

$$K + K - 0 = \lambda \langle x, \xi \rangle = \lambda l(x). \quad (6.14)$$

Hence  $l(x) = 2K/\lambda$ . On the other hand, we get from (6.2) that  $K = \lambda r(x)$  so that  $r(x) = K/\lambda$ . Thus the position vector field  $x$  along  $M$  is given by

$$x = \frac{K}{\lambda}(\xi + 2N). \quad (6.15)$$

Therefore, we obtain

$$\xi = \bar{\nabla}_\xi x = \frac{\xi \cdot K}{\lambda}(\xi + 2N) + \frac{K}{\lambda}(\bar{\nabla}_\xi \xi + \bar{\nabla}_\xi N). \quad (6.16)$$

The last term in this equality is zero ( $\bar{\nabla}_\xi \xi = 0$ ,  $\bar{\nabla}_\xi N = 0$ ). Then use (6.5) to conclude that  $\xi = \xi + 2N$ , that is,  $N \equiv 0$ , which is a contradiction. Hence  $\lambda = 0$ , and the proof is complete.  $\square$

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