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# Twisted Grosse–Wulkenhaar $\phi^{\star 4}$ model: dynamical noncommutativity and Noether currents

**M N Hounkonnou and D Ousmane Samary**

International Chair of Mathematical Physics and Applications (ICMPA-UNESCO Chair), University of Abomey-Calavi, 072 B.P. 50 Cotonou, Republic of Benin

E-mail: norbert.hounkonnou@cipma.uac.bj,hounkonnou@yahoo.fr and ousmanesamarydine@yahoo.fr,samarydine@gmail.com

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**Abstract**

This paper addresses the computation of Noether currrents for the renormalizable Grosse–Wulkenhaar $\phi^{\star 4}$ model subjected to a dynamical noncomutativity realized through a twisted Moyal product. The noncommutative energy–momentum tensor, angular momentum tensor and the dilatation current are explicitly derived. The breaking of translation and rotation invariances has been avoided via a constraint equation.

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## 1. Introduction

Most of the different settings for noncommutative (NC) field theories [1–19] are based on a Moyal space $\mathbb{R}^D_\Theta$, a deformed $D$-dimensional space endowed with a constant Moyal $\star$−bracket of coordinate functions

$$[x^\mu, x^\nu]_\star = \mathrm{i}\Theta^{\mu\nu}, \tag{1}$$

where $\Theta$ is a $D \times D$ non-degenerate skew-symmetric matrix (which requires $D$ even), usually chosen in the form

$$\Theta = \begin{pmatrix} 0 & \Theta_1 & & & & & & \\ -\Theta_1 & 0 & & & 0 & & & \\ & & 0 & \Theta_2 & & & & \\ & & -\Theta_2 & 0 & & & & \\ & & & & \vdots\ \vdots & & & \\ & & & & \vdots\ \vdots & & & \\ & & 0 & & & & 0 & \Theta_{\frac{D}{2}} \\ & & & & & & -\Theta_{\frac{D}{2}} & 0 \end{pmatrix}, \tag{2}$$

where $\Theta_j \in \mathbb{R}$, $j = 1, 2, \ldots, \frac{D}{2}$, have the dimension[1] of length square, ($[\Theta_j] = [L]^2$), and $D$ denotes the spacetime dimension. The corresponding product of functions is the associative, noncommutative Moyal–Groenewold–Weyl product, simply called hereafter the Moyal product or the $\star$-product defined by

$$(f \star g)(x) = m\left\{e^{i\frac{\Theta^{\rho\sigma}}{2}\partial_\rho\otimes\partial_\sigma} f(x)\otimes g(x)\right\} x \in \mathrm{R}^D_\Theta \qquad \forall f, g \in \mathcal{S}(\mathrm{R}^D_\Theta), \tag{3}$$

where $m$ is the ordinary multiplication of functions and $\mathcal{S}(\mathrm{R}^D_\Theta)$, the space of suitable Schwartzian functions. For more details, see [11–14]. Such a noncommutative geometry possesses the specific pathology to break both the Lorentz invariance by the presence of $\Theta^{\mu\nu}$, as $[x^\mu, x^\nu]_\star = i\Theta^{\mu\nu}$ is not generally invariant under rotation, and the local character of the theory due to infinite time derivatives. There result energy momentum tensors (EMTs) which are not locally conserved, not traceless in the massless situation and neither symmetric nor gauge invariant in gauge theories. A number of works exist in attempts to achieve regularization for the NC EMT which then becomes symmetric albeit not locally conserved. Further improvement of this quantity by usual algebraic tricks breaks its symmetry (see [13] and references therein). Therefore, the property of nonlocal conservation of angular momentum is not *a priori* proscribed.

Recently, Paolo Aschieri *et al* [1] introduced a so-called dynamical noncommutativity to investigate Noether currents in an ordinary nonrenormalizable twisted $\phi^{\star 4}$ theory.

This work addresses questions of the applicability of such a formalism on the new class of renormalizable NC field theories built on the Grosse and Wulkenhaar (GW) $\phi^{\star 4}$ scalar field model defined in Euclidean spacetime by the action functional [12]

$$S^\Omega_\star[\phi] = \int \mathrm{d}^D x \left(\frac{1}{2}\partial_\mu\phi\star\partial^\mu\phi + \frac{\Omega^2}{2}(\tilde{x}_\mu\phi)\star(\tilde{x}^\mu\phi) + \frac{m^2}{2}\phi\star\phi + \frac{\lambda}{4!}\phi\star\phi\star\phi\star\phi\right), \tag{4}$$

where $\tilde{x}_\mu = 2(\Theta^{-1})_{\mu\nu}x^\nu$ and $S^\Omega_\star[\phi]$ is covariant under Langmann–Szabo duality [19]. $\Omega$ and $\lambda$ are dimensionless parameters.

The Moyal $\star$-product (3) can be generalized as

$$(f \star g)(x) = m\left\{e^{i\frac{\Theta^{ab}}{2}X_a\otimes X_b} f(x)\otimes g(x)\right\}, \tag{5}$$

where $X_a = e^\mu_a(x)\partial_\mu$ are the commuting vector fields, the index $a$ being just a label for the vector fields. The commutation relation becomes $[x^\mu, x^\nu]_\star = i\Theta^{ab}e^\mu_a(x)e^\nu_b(x) =: i\widetilde{\Theta}^{\mu\nu}(x)$, engendering a twisted scalar field theory where $e^\mu_a$ and hence the $\star$ product itself appear dynamical. The condition $[X_a, X_b] = 0$ implies constraints on $e^\mu_a$, namely $e^\nu_{[a}\partial_\nu e^\mu_{b]} = 0$, that can be solved off-shell in terms of $D$ scalar fields $\phi^a$ (see [1, 2]). Supposing that the square matrix $e^\mu_a$ has an inverse $e^a_\mu$ everywhere, so that the $X_a$ are linearly independent, then the above condition becomes $\partial_{[\mu}e^a_{\nu]} = 0$ which is satisfied by $e^a_\nu = \partial_\nu\phi^a$. Since $X_a\phi^b = \delta^b_a$, the field $\phi^b$ can be seen as new coordinates along the $X_a$ directions. Besides, the Leibniz rule extends to the commuting fields $X_a$ as follows: $X_a(f \star g) = (X_a f)\star g + f \star (X_a g)$.

Furthermore, expanding the dynamical $\star$-product (5) of two functions as

$$\begin{aligned} f \star g &= fg + \frac{i}{2}\Theta^{ab}X_a f X_b g \\ &\quad + \frac{1}{2!}\left(\frac{i}{2}\right)^2 \Theta^{a_1 b_1}\Theta^{a_2 b_2}(X_{a_1}X_{a_2}f)(X_{b_1}X_{b_2}g) + \cdots \\ &\equiv e^\Delta(f, g), \end{aligned} \tag{6}$$

[1] Units such that $\hbar = 1 = c$ are used throughout.

where powers of the bilinear operator $\Delta$ are defined as

$$\Delta(f,g) = \frac{\mathrm{i}}{2}\Theta^{ab}(X_a f)(X_b g) \qquad \Delta^0(f,g) = fg$$
$$\Delta^n(f,g) = \left(\frac{\mathrm{i}}{2}\right)^n \Theta^{a_1b_1}\cdots\Theta^{a_nb_n}(X_{a_1}\cdots X_{a_n} f)(X_{b_1}\cdots X_{b_n} g), \tag{7}$$

one can deduce the following rules (straightforwardly generalizing the usual Moyal product identities):

$$f\star g = fg + X_a T(\Delta)(f, \widetilde{X}^a g) \tag{8}$$
$$[f,g]_\star = f\star g - g\star f = 2X_a S(\Delta)(f,\widetilde{X}^a g) \tag{9}$$
$$\{f,g\}_\star = f\star g + g\star f = 2fg + 2X_a R(\Delta)(f,\widetilde{X}^a g), \tag{10}$$

where

$$T(\Delta) = \frac{e^\Delta - 1}{\Delta} \qquad S(\Delta) = \frac{\sinh(\Delta)}{\Delta}$$
$$R(\Delta) = \frac{\cosh(\Delta)-1}{\Delta} \qquad \text{and} \qquad \widetilde{X}^a = \frac{\mathrm{i}}{2}\Theta^{ab}X_b. \tag{11}$$

$S(\Delta)(\cdot,\widetilde{X}\cdot)$ is a bilinear antisymmetric operator and

$$T(\Delta)(f,\widetilde{X}^a g) - T(\Delta)(g,\widetilde{X}^a f) = 2S(\Delta)(f,\widetilde{X}^a g). \tag{12}$$

This paper is organized as follows. In section 2, we derive the field equations of motion and provide with the explicit computation of relevant physical quantities such as the noncommutative energy momentum tensor (NC EMT), the angular momentum tensor (AMT) and the dilatation current (DC). Furthermore, in section 3, we proceed to the symmetry analysis including the translation, rotation and dilatation transformations and compute the conserved currents. Finally, we end with some concluding remarks in section 4.

## 2. Twisted Grosse–Wulkenhaar model: Noether currents

The integral in (4), defined with the dynamical Moyal $\star$-product (5), is not cyclic; even with suitable boundary conditions at infinity,

$$\int \mathrm{d}^D x\,(f\star g) \neq \int \mathrm{d}^D x\,(g\star f). \tag{13}$$

Using now the measure $ed^{Dx}$ where $e = \det\left(e^a_\mu\right)$, a cyclic integral can be defined so that, up to boundary terms

$$\int e\,\mathrm{d}^D x\,(f\star g) = \int e\,\mathrm{d}^D x(fg) = \int e\,\mathrm{d}^D x\,(g\star f). \tag{14}$$

Therefore, the NC GW Lagrangian action (4) can be rewritten by means of a cyclic integral as follows:

$$\mathcal{S}^\Omega_\star[\phi] =: \int e\mathrm{d}^D x\left(\mathcal{L}^\Omega_\star \star e^{-1}\right)$$
$$=: \int e\,\mathrm{d}^D x\left\{\frac{1}{2}\partial_\mu\phi\star\partial^\mu\phi + \frac{m^2}{2}\phi\star\phi + \frac{\lambda}{4!}\phi\star\phi\star\phi\star\phi + \frac{\Omega^2}{2}(\tilde{x}_\mu\phi)\star(\tilde{x}^\mu\phi) + \frac{1}{2}\partial_\mu\phi_a\star\partial^\mu\phi^a\right\}\star e^{-1} \tag{15}$$

where $e = \det e^a_\mu$. From (15) the peculiar Euler Lagrange equations of motion can be readily derived by direct application of the variational principle and the use of formulas of derivatives and variations given by [1]

$$\begin{aligned}
&\delta_{\phi^c} e^\mu_a = -e^\nu_a e^\mu_b \delta_{\phi^c} e^b_\nu = -e^\nu_a e^\mu_b \partial_\nu \delta\phi^b = -e^\mu_b X_a(\delta\phi^b) \qquad \partial_\mu e = e X_a(\partial_\mu \phi^a)\\
&\delta_{\phi^c} X_a = \delta_{\phi^c}\left(e^\mu_a \partial_\mu\right) = -e^\mu_b X_a(\delta\phi^b)\partial_\mu = -X_a(\delta\phi^b) X_b \qquad (16)\\
&\delta_{\phi^c} e = e X_a(\delta\phi^a) \qquad \delta_{\phi^c} e^{-1} = -e^{-1} X_a(\delta\phi^a) \qquad e X_a(f) = \partial_\mu\left(e e^\mu_a f\right).
\end{aligned}$$

To compute $\delta_{\phi^c}$ variations, it turns out that the identity

$$\delta_{\phi^c}(f \star g) = -(\delta\phi^c X_c f) \star g - f \star (\delta\phi^c X_c g) + \delta\phi^c X_c (f \star g), \qquad (17)$$

where the functions $f$ and $g$ do not depend on $\phi^c$, is useful. By induction, one can immediately prove that (17) holds for $\star$-products of an arbitrary number of factors:

$$\begin{aligned}
\delta_{\phi^c}(f \star g \star \cdots \star h) = &-(\delta\phi^c X_c f) \star g \star \cdots \star h\\
&- f \star (\delta\phi^c X_c g) \star \cdots \star h\\
&- \cdots - f \star g \star \cdots \star (\delta\phi^c X_c h)\\
&+ \delta\phi^c X_c (f \star g \star \cdots \star h). \qquad (18)
\end{aligned}$$

### *2.1. Equations of motion for $\phi$ and related currents*

The action (15) can be viewed as the sum of four actions pertaining to different terms as follows:

$$\begin{aligned}
\mathcal{S}^{\Omega,0}_\star[\phi] &= \frac{1}{2}\int e\, \mathrm{d}^D x (\partial_\mu \phi \star \partial^\mu \phi \star + \partial_\mu \phi_a \star \partial^\mu \phi^a) \star e^{-1}\\
\mathcal{S}^{\Omega,m^2}_\star[\phi] &= \frac{m^2}{2}\int e\, \mathrm{d}^D x (\phi \star \phi \star e^{-1})\\
\mathcal{S}^{\Omega,\lambda}_\star[\phi] &= \frac{\lambda}{4!}\int e\, \mathrm{d}^D x (\phi \star \phi \star \phi \star \phi \star e^{-1})\\
\mathcal{S}^{\Omega,\mathrm{har}}_\star[\phi] &= \frac{\Omega^2}{2}\int e\, \mathrm{d}^D x (\tilde{x}_\mu \phi) \star (\tilde{x}^\mu \phi) \star e^{-1}.
\end{aligned}$$

The variations of these quantities with respect to the field $\phi$, using the cyclicity property of the integral, give the following relations:

$$\begin{aligned}
\delta_\phi(\mathcal{S}^{\Omega,0}_\star) = \int \mathrm{d}^D x \Bigg\{ &-\delta\phi . \partial_\sigma \left(\frac{e}{2}\{\partial^\sigma \phi, e^{-1}\}_\star\right) + \partial_\sigma \Bigg[\frac{e\delta\phi}{2}.\{\partial^\sigma \phi, e^{-1}\}_\star + e e^\sigma_b T(\Delta)\\
&\times \left(\delta \partial_\mu \phi, \frac{\widetilde{X}^b}{2}\{\partial^\mu \phi, e^{-1}\}_\star\right) + e e^\sigma_b S(\Delta)(\partial_\mu \phi, \widetilde{X}^b(\partial^\mu \delta\phi \star e^{-1}))\Bigg]\Bigg\} \qquad (19)
\end{aligned}$$

$$\begin{aligned}
\delta_\phi(\mathcal{S}^{\Omega,m^2}_\star) = \int \mathrm{d}^D x \Bigg\{ &\delta\phi \cdot \left(\frac{e m^2}{2}\{\phi, e^{-1}\}_\star\right) + \partial_\sigma \Bigg[\frac{m^2}{2} e e^\sigma_b T(\Delta)(\delta\phi, \widetilde{X}^b\{\phi, e^{-1}\}_\star)\\
&+ m^2 e e^\sigma_b S(\Delta)(\phi, \widetilde{X}^b(\delta\phi \star e^{-1}))\Bigg]\Bigg\} \qquad (20)
\end{aligned}$$

$$\begin{aligned}
\delta_\phi(\mathcal{S}^{\Omega,\lambda}_\star) = \frac{\lambda}{4!}\int \mathrm{d}^D x \big\{ &\delta\phi\{\phi \star \phi, \{\phi, e^{-1}\}_\star\}_\star + \partial_\sigma \big[e e^\sigma_b T(\Delta)(\delta\phi, \widetilde{X}^b\{\phi \star \phi, \{\phi, e^{-1}\}_\star\}_\star)\\
&+ 2 e e^\sigma_b S(\Delta)(\phi, \widetilde{X}^b(\delta\phi \star \phi \star \phi \star e^{-1})) + 2 e e^\sigma_b S(\Delta)(\phi \star \phi, \widetilde{X}^b(\delta\phi \star \phi \star e^{-1}))\\
&+ 2 e e^\sigma_b S(\Delta)(\phi \star \phi \star \phi, \widetilde{X}^b(\delta\phi \star e^{-1}))\big]\big\} \qquad (21)
\end{aligned}$$

$$\delta_\phi\big(\mathcal{S}_\star^{\Omega,\text{har}}\big) = \frac{\Omega^2}{8}\int \mathrm{d}^D x\big\{e\delta\phi\{\tilde{x},\{e^{-1},\{\tilde{x},\phi\}_\star\}_\star\}_\star + \partial_\sigma\big[ee_b^\sigma T(\Delta)(\delta\phi,\widetilde{X}^b\{\tilde{x},\{e^{-1},\{\tilde{x},\phi\}_\star\}_\star\}_\star)$$
$$+ 2ee_b^\sigma S(\Delta)(\tilde{x},\widetilde{X}^b(\delta\phi\star\{\tilde{x},\phi\}_\star\star e^{-1})) + 2ee_b^\sigma S(\Delta)(\{\tilde{x},\phi\star\tilde{x}\}_\star, X^b(\delta\phi\star e^{-1}))$$
$$+ 2ee_b^\sigma S(\Delta)(\{\phi,\tilde{x}\}_\star,\widetilde{X}^b(\delta\phi\star\tilde{x}\star e^{-1}))\big]\big\}. \tag{22}$$

Summing all these four variations and factoring out $\delta\phi$ from the resulting expression and grouping the surface terms, source of the current hereafter denoted by $\mathcal{K}^\sigma$, we can write the GW action variation with respect to the field $\phi$ into the global form

$$\delta_\phi\mathcal{S}_\star^\Omega = \int \mathrm{d}^D x\,(\delta\phi\mathcal{E}_\phi + \partial_\sigma\mathcal{K}^\sigma) \tag{23}$$

from which we deduce the equation of motion of the field $\phi$ as

$$\mathcal{E}_\phi = -\frac{1}{2}\partial_\sigma(e\{\partial^\sigma\phi,e^{-1}\}_\star) + \frac{m^2}{2}e\{\phi,e^{-1}\}_\star + \frac{\lambda}{4!}e\{\phi\star\phi,\{\phi,e^{-1}\}_\star\}_\star$$
$$+ \frac{\Omega^2}{8}e\{\tilde{x},\{e^{-1},\{\tilde{x},\phi\}_\star\}_\star\}_\star = 0. \tag{24}$$

In the commutative limit $\Theta\to 0$, equation (24) becomes the usual $\phi^4$ field equation of motion

$$\Box\phi - m^2\phi - \frac{\lambda}{3!}\phi^3 = 0. \tag{25}$$

The current $\mathcal{K}^\sigma$ results from the combination of the contributions

$$\mathcal{K}^\sigma = \mathcal{K}^\sigma(0) + \mathcal{K}^\sigma(m^2) + \mathcal{K}^\sigma(\lambda) + \mathcal{K}^\sigma(\Omega^2) \tag{26}$$

induced, respectively, by

(i) the velocity term contribution

$$\mathcal{K}^\sigma(0) = \frac{e\delta\phi}{2}.\{\partial^\sigma\phi,e^{-1}\}_\star + ee_b^\sigma\bigg[T(\Delta)\bigg(\delta\partial_\mu\phi,\frac{\widetilde{X}^b}{2}\{\partial^\mu\phi,e^{-1}\}_\star\bigg)$$
$$+ S(\Delta)(\partial_\mu\phi,\widetilde{X}^b(\partial^\mu\delta\phi\star e^{-1}))\bigg], \tag{27}$$

(ii) the mass term

$$\mathcal{K}^\sigma(m^2) = ee_b^\sigma\bigg[\frac{m^2}{2}T(\Delta)(\delta\phi,\widetilde{X}^b\{\phi,e^{-1}\}_\star) + m^2 S(\Delta)(\phi,\widetilde{X}^b(\delta\phi\star e^{-1}))\bigg], \tag{28}$$

(iii) the $\phi^{\star 4}$ interaction

$$\mathcal{K}^\sigma(\lambda) = ee_b^\sigma\bigg[\frac{\lambda}{4!}T(\Delta)(\delta\phi,\widetilde{X}^b\{\phi\star\phi,\{\phi,e^{-1}\}_\star\}_\star) + \frac{\lambda}{12}S(\Delta)(\phi,\widetilde{X}^b(\delta\phi\star\phi\star\phi\star e^{-1}))$$
$$+ \frac{\lambda}{12}S(\Delta)(\phi\star\phi,\widetilde{X}^b(\delta\phi\star\phi\star e^{-1}))$$
$$+ \frac{\lambda}{12}S(\Delta)(\phi\star\phi\star\phi,\widetilde{X}^b(\delta\phi\star e^{-1}))\bigg], \tag{29}$$

(iv) the GW harmonic interaction

$$\mathcal{K}^\sigma(\Omega^2) = ee_b^\sigma\bigg[\frac{\Omega^2}{8}T(\Delta)(\delta\phi,\widetilde{X}^b\{\tilde{x},\{e^{-1},\{\tilde{x},\phi\}_\star\}_\star\}_\star)$$
$$+ \frac{\Omega^2}{4}S(\Delta)(\tilde{x},\widetilde{X}^b(\delta\phi\star\{\tilde{x},\phi\}_\star\star e^{-1}))$$
$$+ \frac{\Omega^2}{4}S(\Delta)(\{\tilde{x},\phi\star\tilde{x}\}_\star, X^b(\delta\phi\star e^{-1}))$$
$$+ \frac{\Omega^2}{4}S(\Delta)(\{\phi,\tilde{x}\}_\star,\widetilde{X}^b(\delta\phi\star\tilde{x}\star e^{-1}))\bigg]. \tag{30}$$

*2.2. Equations of motion for $\phi^c$ and related currents*

The $\phi^c$ variation of the action (15)

$$\delta_{\phi^c} S = \delta_{\phi^c}\left\{\int \mathrm{d}^D x e\big[\mathcal{L}^{\Omega}_{\star} \star e^{-1}\big]\right\}$$
$$= \int \mathrm{d}^D x\big[(\delta_{\phi^c} e)\mathcal{L}^{\Omega}_{\star} \star e^{-1} + e\big(\delta_{\phi^c}(\mathcal{L}^{\Omega}_{\star} \star e^{-1})\big)\big], \tag{31}$$

where the ordinary Leibniz rule is used when the variation $\delta_{\phi^c}$ acts on the pointwise product, can be considered as a sum of two terms $A$ and $B$. The term $A$ is given by

$$A = \int \mathrm{d}^D x(\delta_{\phi^c} e)\big[\mathcal{L}^{\Omega}_{\star} \star e^{-1}\big]$$
$$= \int \mathrm{d}^D x\big[-e\delta\phi^a X_a(\mathcal{L}^{\Omega}_{\star} \star e^{-1}) + \partial_\rho\big(ee_a{}^{\rho}\delta\phi^a(\mathcal{L}^{\Omega}_{\star} \star e^{-1})\big)\big], \tag{32}$$

where $\delta_{\phi^c} e = eX_a(\delta\phi^a)$, while the second term $B$ encompasses contributions from the velocity, the mass, the $\phi^4$ interaction and the harmonic potential denoted by $B_0$, $B_{m^2}$, $B_\lambda$, $B_{\text{har}}$, respectively.

The mass term, $B_{m^2}$, depends on the $\star$ product and $e^{-1}$ as follows:

$$B_{m^2} = \frac{m^2}{2}\int \mathrm{d}^D x e\delta_{\phi^c}(\phi \star \phi \star e^{-1})$$
$$= \frac{m^2}{2}\int \mathrm{d}^D x e(-(\delta\phi^a X_a\phi) \star \phi \star e^{-1} - \phi \star (\delta\phi^a X_a\phi) \star e^{-1}$$
$$- \phi \star \phi \star (\delta\phi^a X_a e^{-1}) + \delta\phi^a X_a(\phi \star \phi \star e^{-1}) + \phi \star \phi \star (\delta_{\phi^c} e^{-1})). \tag{33}$$

Noting that $\delta_{\phi^c} e^{-1} = -e^{-1}(X_a\delta\phi^a)$, we obtain

$$B_{m^2} = \frac{m^2}{2}\int \mathrm{d}^D x e\{\delta\phi^a X_a(\phi \star \phi \star e^{-1}) - \phi \star \phi \star X_a(\delta\phi^a e^{-1})$$
$$- (\delta\phi^a X_a\phi) \star \phi \star e^{-1} - \phi \star (\delta\phi^a X_a\phi) \star e^{-1}\}. \tag{34}$$

Adding and subtracting $(\delta\phi^a X_a\phi) \star e^{-1} \star \phi$ from (34) enable us to combine the terms under the integral in the following way:

$$B_{m^2} = \frac{m^2}{2}\int \mathrm{d}^D x\{e\delta\phi^a(X_a(\phi \star \phi \star e^{-1}) + e^{-1}X_a(\phi \star \phi) - (X_a\phi)\{\phi, e^{-1}\}_\star)$$
$$+ eX_b(-\phi \star \phi \star (\delta\phi^b e^{-1}) + T(\Delta)[X_a(\phi \star \phi), \widetilde{X}^b(\delta\phi^a e^{-1})]$$
$$- T(\Delta)[\delta\phi^a(X_a\phi), \widetilde{X}^b\{\phi \overset{\star}{,} e^{-1}\}] + 2S(\Delta)[\delta\phi^a(X_a\phi) \star e^{-1}, \widetilde{X}^b\phi])\}, \tag{35}$$

where the terms proportional to $e\delta\phi^a$

$$\frac{m^2}{2} e\delta\phi^a(X_a(\phi \star \phi \star e^{-1}) + e^{-1}X_a(\phi \star \phi) - (X_a\phi)\{\phi, e^{-1}\}_\star) \tag{36}$$

contribute to the equation of motion, while those proportional to $eX_b$

$$-\frac{m^2}{2}\partial_\mu\big\{ee^\mu_b(-\phi \star \phi \star (\delta\phi^b e^{-1}) + T(\Delta)[X_a(\phi \star \phi), \widetilde{X}^b(\delta\phi^a e^{-1})]$$
$$- T(\Delta)[\delta\phi^a(X_a\phi), \widetilde{X}^b\{\phi, e^{-1}\}_\star] + 2S(\Delta)[\delta\phi^a(X_a\phi) \star e^{-1}, \widetilde{X}^b\phi])\big\} \tag{37}$$

are surface terms giving the current $\mathcal{J}^\sigma$ to be defined later. The remaining contributions can be computed in an analogous way to give the relations

$$\begin{aligned}
B_0 = \frac{1}{2}\int \mathrm{d}^D x\big\{ & e\delta\phi^c[X_c(\partial_\mu\phi \star \partial^\mu\phi \star e^{-1}) + e^{-1}X_c(\partial_\mu\phi \star \partial^\mu\phi) \\
& - X_c\partial_\mu\phi.\{\partial^\mu\phi, e^{-1}\}_\star] + \partial_\sigma\big[-ee^\sigma_b(\partial_\mu\phi \star \partial^\mu\phi \star (\delta\phi^b e^{-1})) \\
& - ee^\sigma_b T(\Delta)(\delta\phi^c X_c\partial_\mu\phi, \widetilde{X}^b\{\partial^\mu\phi, e^{-1}\}_\star) \\
& - 2ee^\sigma_b S(\Delta)(\partial^\mu\phi, \widetilde{X}^b((\delta\phi^c X_c\partial_\mu\phi) \star e^{-1})) \\
& + ee^\sigma_b T(\Delta)(X_c(\partial_\mu\phi \star \partial^\mu\phi), \widetilde{X}^b(\delta\phi^c e^{-1}))\big]\big\} \\
& + \frac{1}{2}\int \mathrm{d}^D x\left\{ e\delta\phi^c\left[ X_c(\partial_\mu\phi_a \star \partial^\mu\phi^a \star e^{-1}) + e^{-1}X_c(\partial_\mu\phi_a \star \partial^\mu\phi^a)\right.\right. \\
& \left. - X_c\partial_\mu\phi_a.\{\partial^\mu\phi^a, e^{-1}\}_\star - \frac{2}{e}\partial_\mu\left(\frac{e}{2}\{\partial^\mu\phi_c, e^{-1}\}_\star\right)\right] \\
& + \partial_\sigma\Big[-ee^\sigma_b(\partial_\mu\phi_a \star \partial^\mu\phi^a \star (\delta\phi^b e^{-1})) + e\delta\phi^a\{\partial^\sigma\phi_a, e^{-1}\}_\star \\
& - ee^\sigma_b T(\Delta)(\delta\phi^c X_c\partial_\mu\phi_a, \widetilde{X}^b\{\partial^\mu\phi^a, e^{-1}\}_\star) \\
& - 2ee^\sigma_b S(\Delta)(\partial^\mu\phi_a, \widetilde{X}^b((\delta\phi^c X_c\partial_\mu\phi^a) \star e^{-1})) \\
& + 2ee^\sigma_b S(\Delta)(\partial_\mu\phi_a, \widetilde{X}^b(\partial^\mu\delta\phi^a \star e^{-1})) \\
& + 2ee^\sigma_b T(\Delta)\left(\delta\partial_\mu\phi_a, \frac{\widetilde{X}^b}{2}\{\partial^\mu\phi^a, e^{-1}\}_\star\right) \\
& \left.+ ee^\sigma_b T(\Delta)(X_c(\partial_\mu\phi_a \star \partial^\mu\phi^a), \widetilde{X}^b(\delta\phi^c e^{-1}))\Big]\right\}
\end{aligned} \tag{38}$$

$$\begin{aligned}
B_\lambda = \frac{\lambda}{4!}\int \mathrm{d}^D x\big\{ & e\delta\phi^a(X_a(\phi \star \phi \star \phi \star \phi \star e^{-1}) + e^{-1}X_a(\phi \star \phi \star \phi \star \phi) \\
& - X_a\phi.\{\phi \star \phi, \{\phi, e^{-1}\}_\star\}_\star) + \partial_\sigma\big[-ee^\sigma_b(\phi \star \phi \star \phi \star \phi \star \delta\phi^b e^{-1}) \\
& - ee^\sigma_b T(\Delta)(\delta\phi^c X_c\phi, \widetilde{X}^b\{\phi \star \phi, \{\phi, e^{-1}\}_\star\}_\star) \\
& - 2ee^\sigma_b S(\Delta)(\phi, \widetilde{X}^b((\delta\phi^c X_c\phi) \star \phi \star \phi \star e^{-1})) \\
& - 2ee^\sigma_b S(\Delta)(\phi \star \phi, \widetilde{X}^b((\delta\phi^c X_c\phi) \star \phi \star e^{-1})) \\
& - 2ee^\sigma_b S(\Delta)(\phi \star \phi \star \phi, \widetilde{X}^b((\delta\phi^c X_c\phi) \star e^{-1})) \\
& + ee^\sigma_b T(\Delta)(X_c(\phi \star \phi \star \phi \star \phi), \widetilde{X}^b(\delta\phi^c e^{-1}))\big]\big\}
\end{aligned} \tag{39}$$

$$\begin{aligned}
B_{\mathrm{har}} = \frac{\Omega^2}{2}\int \mathrm{d}^D x\big\{ & e\delta\phi^c(X_c((\tilde{x}\phi) \star (\tilde{x}\phi) \star e^{-1}) + e^{-1}X_c((\tilde{x}\phi) \star (\tilde{x}\phi)) \\
& - \phi X_c\tilde{x}.\{\tilde{x}\phi, e^{-1}\}_\star - (X_c\phi)\tilde{x}.\{\tilde{x}\phi, e^{-1}\}_\star) \\
& + \partial_\sigma\big[-ee^\sigma_b((\tilde{x}\phi) \star (\tilde{x}\phi) \star (\delta\phi^b e^{-1})) - ee^\sigma_b T(\Delta)(\delta\phi^c X_c(\tilde{x}\phi), \widetilde{X}^b\{\tilde{x}\phi, e^{-1}\}_\star) \\
& - 2ee^\sigma_b S(\Delta)(\tilde{x}\phi, \widetilde{X}^b((\delta\phi^c X_c(\tilde{x}\phi)) \star e^{-1})) \\
& + ee^\sigma_b T(\Delta)(X_c((\tilde{x}\phi) \star (\tilde{x}\phi)), \widetilde{X}^b(\delta\phi^c e^{-1}))\big]\big\}.
\end{aligned} \tag{40}$$

Summing now all the contributions and rearranging, after tedious algebraic transformations, in terms of two components representing the $\delta\phi^c$ factor and the current surface counterpart, the GW action $\phi^c$-variation takes the form

$$\delta_{\phi^c} \mathcal{S}_\star^\Omega = \int \mathrm{d}^D x (\delta\phi^c \mathcal{E}_{(\phi,\phi^c)} + \partial_\sigma \mathcal{J}^\sigma) \quad (41)$$

$$= \int \mathrm{d}^D x (-\delta\phi^c X_c \phi \mathcal{E}_\phi - \delta\phi^c \mathcal{E}_{\phi^c} + \partial_\sigma \mathcal{J}^\sigma) \quad (42)$$

from which we get the following field equation:

$$\mathcal{E}_{(\phi,\phi^c)} = e\left[\frac{1}{e} X_c(\mathcal{L}_\star^\Omega) - (X_c\phi)\left(\frac{m^2}{2}\{\phi, e^{-1}\}_\star + \frac{\lambda}{4!}\{\phi\star\phi, \{\phi, e^{-1}\}_\star\}_\star + \frac{\Omega^2}{2}\tilde{x}.\{\tilde{x}\phi, e^{-1}\}_\star\right)\right.$$
$$- \frac{\Omega^2}{2}\phi X_c \tilde{x}.\{\tilde{x}\phi, e^{-1}\}_\star - \frac{1}{2}X_c\partial_\mu\phi.\{\partial^\mu\phi, e^{-1}\}_\star - \frac{1}{2}X_c\partial_\mu\phi_a.\{\partial^\mu\phi^a, e^{-1}\}_\star$$
$$\left. - \frac{1}{e}\partial_\mu\left(\frac{e}{2}\{\partial^\mu\phi_c, e^{-1}\}_\star\right)\right] = 0. \quad (43)$$

Using the identities $\tilde{x}_\mu \star \phi = \tilde{x}_\mu\phi + \mathrm{i}\partial_\mu\phi$ and $\phi\star\tilde{x}_\mu = \tilde{x}_\mu\phi - \mathrm{i}\partial_\mu\phi$ implying $\tilde{x}\phi = \frac{1}{2}\{\tilde{x}, \phi\}_\star$, we can deduce that $\frac{\Omega^2}{2}\tilde{x}.\{\tilde{x}\phi, e^{-1}\}_\star = \frac{\Omega^2}{8}\{\tilde{x}, \{e^{-1}, \{\tilde{x}, \phi\}_\star\}_\star\}_\star$, and the equation of motion can be re-expressed as

$$\mathcal{E}_{(\phi,\phi^c)} = -X_c\phi\mathcal{E}_\phi + X_c\mathcal{L}_\star^\Omega - \frac{1}{2}X_c\phi\partial_\mu(e\{\partial^\mu\phi, e^{-1}\}_\star) - e\frac{\Omega^2}{2}\phi X_c\tilde{x}.\{\tilde{x}\phi, e^{-1}\}_\star$$
$$- \frac{e}{2}X_c\partial_\mu\phi.\{\partial^\mu\phi, e^{-1}\}_\star - \frac{e}{2}X_c\partial_\mu\phi_a.\{\partial^\mu\phi^a, e^{-1}\}_\star - \partial_\mu\left(\frac{e}{2}\{\partial^\mu\phi_c, e^{-1}\}_\star\right)$$
$$= -X_c\phi\mathcal{E}_\phi - \mathcal{E}_{\phi^c} = 0, \quad (44)$$

where

$$\mathcal{E}_{\phi^c} = -X_c\mathcal{L}_\star^\Omega + \frac{1}{2}X_c\phi\partial_\mu(e\{\partial^\mu\phi, e^{-1}\}_\star) + e\frac{\Omega^2}{2}\phi X_c\tilde{x}.\{\tilde{x}\phi, e^{-1}\}_\star + \frac{e}{2}X_c\partial_\mu\phi.\{\partial^\mu\phi, e^{-1}\}_\star$$
$$+ \frac{e}{2}X_c\partial_\mu\phi_a.\{\partial^\mu\phi^a, e^{-1}\}_\star + \partial_\mu\left(\frac{e}{2}\{\partial^\mu\phi_c, e^{-1}\}_\star\right) \quad (45)$$

with

$$\frac{\Omega^2}{2}\phi X_c\tilde{x}.\{\tilde{x}\phi, e^{-1}\}_\star = \frac{\Omega^2}{8}X_c\tilde{x}.\{\phi, \{e^{-1}, \{\tilde{x}, \phi\}_\star\}_\star\}_\star.$$

One can immediately show that, as expected from [1], when $\phi$ is on shell (i.e. $\mathcal{E}_\phi = 0$), the $\phi^c$ field equation of motion simply reduces to $\mathcal{E}_{\phi^c} = 0$, and in the commutative limit, we get $\Box\phi^c = 0$ as it should. Besides, the field equations (24) and (45) are trivially satisfied by the solution $\phi = 0$, $e^a_\mu = \partial_\mu\phi^a = \delta^a_\mu$ corresponding to the usual Moyal product. The field $\phi$ acts as a source for the noncommutativity field $\phi^c$.

In the same vein, the current $\mathcal{J}^\sigma$ is given by

$$\mathcal{J}^\sigma = \mathcal{J}^\sigma(0) + \mathcal{J}^\sigma(m^2) + \mathcal{J}^\sigma(\lambda) + \mathcal{J}^\sigma(\Omega^2), \quad (46)$$

where the contributions engendered by the velocity term, the mass term, the $\phi^{\star 4}$ interaction and the GW harmonic interaction source are, respectively, expressed as

$$\mathcal{J}^\sigma(0) = \frac{1}{2}e\delta\phi^a\{\partial^\sigma\phi_a, e^{-1}\}_\star + ee^\sigma_b\left\{\frac{1}{2}\left[-T(\Delta)(\delta\phi^c X_c\partial_\mu\phi_a, \widetilde{X}^b\{\partial^\mu\phi^a, e^{-1}\}_\star)\right.\right.$$
$$- 2S(\Delta)(\partial^\mu\phi_a, \widetilde{X}^b((\delta\phi^c X_c\partial_\mu\phi^a)\star e^{-1})) + 2S(\Delta)(\partial_\mu\phi_a, \widetilde{X}^b(\partial^\mu\delta\phi^a\star e^{-1}))$$
$$\left. + 2T(\Delta)\left(\delta\partial_\mu\phi_a, \frac{\widetilde{X}^b}{2}\{\partial^\mu\phi^a, e^{-1}\}_\star\right)\right]$$

$$+\frac{1}{2}\Bigg[-T(\Delta)(\delta\phi^c X_c\partial_\mu\phi,\widetilde{X}^b\{\partial^\mu\phi,e^{-1}\}_\star)$$

$$-2S(\Delta)(\partial^\mu\phi,\widetilde{X}^b((\delta\phi^c X_c\partial_\mu\phi)\star e^{-1}))\Bigg]-\mathcal{L}_\star^\Omega(0)\star(\delta\phi^b e^{-1})$$

$$+\delta\phi^b(\mathcal{L}_\star^\Omega(0)\star e^{-1})+T(\Delta)(X_c(\mathcal{L}_\star^\Omega(0)),\widetilde{X}^b(\delta\phi^c e^{-1}))\Bigg\} \tag{47}$$

$$\mathcal{J}^\sigma(m^2)=ee_b^\sigma\Big\{\frac{m^2}{2}\big[-T(\Delta)(\delta\phi^a(X_a\phi),\widetilde{X}^b\{\phi,e^{-1}\})+2S(\Delta)(\delta\phi^a(X_a\phi)\star e^{-1},\widetilde{X}^b\phi)\big]$$
$$-\mathcal{L}_\star^\Omega(m^2)\star(\delta\phi^b e^{-1})+\delta\phi^b(\mathcal{L}_\star^\Omega(m^2)\star e^{-1})$$
$$+T(\Delta)(X_c(\mathcal{L}_\star^\Omega(m^2)),\widetilde{X}^b(\delta\phi^c e^{-1}))\Big\} \tag{48}$$

$$\mathcal{J}^\sigma(\lambda)=ee_b^\sigma\Big\{\frac{\lambda}{4!}\big[-T(\Delta)(\delta\phi^c X_c\phi,\widetilde{X}^b\{\phi\star\phi,\{\phi,e^{-1}\}_\star\}_\star)$$
$$-2S(\Delta)(\phi,\widetilde{X}^b((\delta\phi^c X_c\phi)\star\phi\star\phi\star e^{-1}))$$
$$-2S(\Delta)(\phi\star\phi,\widetilde{X}^b((\delta\phi^c X_c\phi)\star\phi\star e^{-1}))$$
$$-2S(\Delta)(\phi\star\phi\star\phi,\widetilde{X}^b((\delta\phi^c X_c\phi)\star e^{-1}))\big]$$
$$-\mathcal{L}_\star^\Omega(\lambda)\star(\delta\phi^b e^{-1})+\delta\phi^b(\mathcal{L}_\star^\Omega(\lambda)\star e^{-1})$$
$$+T(\Delta)(X_c(\mathcal{L}_\star^\Omega(\lambda)),\widetilde{X}^b(\delta\phi^c e^{-1}))\Big\} \tag{49}$$

$$\mathcal{J}^\sigma(\Omega^2)=ee_b^\sigma\Bigg\{\frac{\Omega^2}{2}[-T(\Delta)(\delta\phi^c X_c(\tilde{x}\phi),\widetilde{X}^b\{\tilde{x}\phi,e^{-1}\}_\star)$$
$$-2S(\Delta)(\tilde{x}\phi,\widetilde{X}^b((\delta\phi^c X_c(\tilde{x}\phi))\star e^{-1}))]$$
$$-\mathcal{L}_\star^\Omega(\Omega^2)\star(\delta\phi^b e^{-1})+\delta\phi^b(\mathcal{L}_\star^\Omega(\Omega^2)\star e^{-1})$$
$$+T(\Delta)(X_c(\mathcal{L}_\star^\Omega(\Omega^2)),\widetilde{X}^b(\delta\phi^c e^{-1}))\Bigg\}, \tag{50}$$

where

$$\mathcal{L}_\star^\Omega(0)=\frac{1}{2}\partial_\mu\phi\star\partial^\mu\phi+\frac{1}{2}\partial_\mu\phi_a\star\partial^\mu\phi^a \qquad \mathcal{L}_\star^\Omega(m^2)=\frac{m^2}{2}\phi\star\phi$$
$$\mathcal{L}_\star^\Omega(\lambda)=\frac{\lambda}{4!}\phi\star\phi\star\phi\star\phi \qquad \mathcal{L}_\star^\Omega(\Omega^2)=\frac{\Omega^2}{2}(\tilde{x}_\mu\phi)\star(\tilde{x}^\mu\phi). \tag{51}$$

## 3. Symmetries and conserved currents

Let us now deal with the symmetry analysis and deduce the conserved currents. Performing a functional variation of the fields and a coordinate transformation

$$\phi'(x)=\phi(x)+\delta\phi(x) \qquad \phi'^c(x)=\phi^c(x)+\delta\phi^c(x) \qquad x'^\mu=x^\mu+\epsilon^\mu \tag{52}$$

and using $\mathrm{d}^Dx'=[1+\partial_\mu\epsilon^\mu+\mathbf{O}(\epsilon^2)]\mathrm{d}^Dx$ lead to the following variation of the action, to the first order in $\delta\phi(x)$, $\delta\phi^c(x)$, $\tilde{x}$ and $\epsilon^\mu$:

$$\begin{aligned}\delta\mathcal{S}_\star^\Omega &= \int e\,\mathrm{d}^Dx\left\{\left|\frac{\partial x'}{\partial x}\right|\star(\mathcal{L}'^{\Omega}_\star\star e^{-1})\right\} - \int e\,\mathrm{d}^Dx\,(\mathcal{L}^\Omega_\star\star e^{-1})\\ &= \int \mathrm{d}^Dx\big\{\delta\big((\mathcal{L}^\Omega_\star\star e^{-1})e\big)+\partial_\mu\epsilon^\mu\star\big((\mathcal{L}^\Omega_\star\star e^{-1})e\big)\big\}\\ &= \int \mathrm{d}^Dx\big\{\delta_\phi\big((\mathcal{L}^\Omega_\star\star e^{-1})e\big)+\delta_{\phi^c}\big((\mathcal{L}^\Omega_\star\star e^{-1})e\big)+\delta_{\tilde{x}}\big(\mathcal{L}^\Omega_\star\star e^{-1}\big)e\big)\\ &\quad+\epsilon^\mu\star\partial_\mu\big[(\mathcal{L}^\Omega_\star\star e^{-1})e\big]+\partial_\mu\epsilon^\mu\star\big(\mathcal{L}^\Omega_\star\star e^{-1}\big)e\big\}. \end{aligned} \tag{53}$$

On shell, and integrated on a submanifold $M\subset\mathbb{R}^D$ with fields nonvanishing at the boundary (so that the total derivative terms do not disappear), we get

$$\delta\mathcal{S}_\star^\Omega = \int_M \mathrm{d}^Dx\,\partial_\sigma\big[\mathcal{K}^\sigma+\mathcal{J}^\sigma+\mathcal{R}^\sigma+\epsilon^\sigma\star\big((\mathcal{L}^\Omega_\star\star e^{-1})e\big)\big] \tag{54}$$

encompassing different contributions explicated below. In the computations, we decompose the GW harmonic term as follows [13]:

$$(\tilde{x}\phi)\star(\tilde{x}\phi) = \tfrac{1}{4}(\tilde{x}\star\phi\star\tilde{x}\star\phi+\tilde{x}\star\phi\star\phi\star\tilde{x}+\phi\star\tilde{x}\star\tilde{x}\star\phi+\phi\star\tilde{x}\star\phi\star\tilde{x}) \tag{55}$$

in order to get the NC Lagrangian entirely lying in the $\star$-algebra of fields with the advantage to be stable under formal $\star$-algebraic computations (such that the cyclicity of $\star$-factors under integral). By first performing the $\phi^c$ variation of the harmonic term in the GW action (15), using the right-hand side of (55), and then identifying the result with (40) one can infer the identity

$$\begin{aligned}&\frac{\Omega^2}{2}\{-T(\Delta)(\delta\phi^cX_c(\tilde{x}\phi),\widetilde{X}^b\{\tilde{x}\phi,e^{-1}\}_\star)\}+\frac{\Omega^2}{2}\{-2S(\Delta)(\tilde{x}\phi,\widetilde{X}^b((\delta\phi^cX_c(\tilde{x}\phi))\star e^{-1}))\}\\ &\quad=\frac{\Omega^2}{8}\{-T(\Delta)(\delta\phi^cX_c\phi,\widetilde{X}^b(\{\tilde{x},\{e^{-1},\{\tilde{x},\phi\}_\star\}_\star\}_\star))\\ &\quad-T(\Delta)(\delta\phi^cX_c\tilde{x},\widetilde{X}^b(\{\phi,\{e^{-1},\{\tilde{x},\phi\}_\star\}_\star\}_\star))\}\\ &\quad+\frac{\Omega^2}{4}\{-S(\Delta)(\tilde{x},\widetilde{X}^b((\delta\phi^cX_c\phi)\star\{\phi,\tilde{x}\}_\star\star e^{-1}))\\ &\quad-S(\Delta)(\{\tilde{x},\phi\star\tilde{x}\}_\star,\widetilde{X}^b((\delta\phi^cX_c\phi)\star e^{-1}))\\ &\quad-S(\Delta)(\{\phi,\tilde{x}\}_\star,\widetilde{X}^b((\delta\phi^cX_c\phi)\star\tilde{x}\star e^{-1}))\\ &\quad-S(\Delta)(\phi,\widetilde{X}^b((\delta\phi^cX_c\tilde{x})\star\{\phi,\tilde{x}\}_\star\star e^{-1}))\\ &\quad-S(\Delta)(\{\phi,\tilde{x}\}_\star,\widetilde{X}^b((\delta\phi^cX_c\tilde{x})\star\phi\star e^{-1}))\\ &\quad-S(\Delta)(\{\phi,\tilde{x}\star\phi\}_\star,\widetilde{X}^b((\delta\phi^cX_c\tilde{x})\star e^{-1}))\}.\end{aligned} \tag{56}$$

Then, from the $\tilde{x}$ variation of the action (15) expressed as

$$\delta_{\tilde{x}}\mathcal{S}_\star^\Omega = \int \mathrm{d}^Dx\left(\delta\tilde{x}\,e\frac{\Omega^2}{8}\{\phi,\{e^{-1},\{\tilde{x},\phi\}_\star\}_\star\}_\star+\partial_\sigma\mathcal{R}^\sigma\right), \tag{57}$$

we deduce the current term

$$\begin{aligned}\mathcal{R}^\sigma = \frac{\Omega^2}{8}ee^\sigma_b\{&T(\Delta)(\delta\tilde{x},\widetilde{X}^b\{\phi,\{e^{-1},\{\tilde{x},\phi\}_\star\}_\star\}_\star)+2S(\Delta)(\{\tilde{x},\phi\}_\star,\widetilde{X}^b(\delta\tilde{x}\star\phi\star e^{-1}))\\ &+2S(\Delta)(\{\phi,\tilde{x}\star\phi\}_\star,\widetilde{X}^b(\delta\tilde{x}\star e^{-1}))+2S(\Delta)(\phi,\widetilde{X}^b(\delta\tilde{x}\star\{\tilde{x},\phi\}_\star\star e^{-1}))\}.\end{aligned} \tag{58}$$

On the other side, using the identity $\delta\phi^c X_c\partial_\mu\phi = \partial_\mu(\delta\phi^c X_c\phi) - \partial_\mu(\delta\phi^c e_c^\rho)\partial_\rho\phi$ and (56), and collecting different terms in an appropriate way, the current $\mathcal{J}^\sigma$ (46) can be now written as

$$\begin{aligned}\mathcal{J}^\sigma &= \mathcal{K}^\sigma(\delta\phi \to -\delta\phi^c X_c\phi) + \mathcal{R}^\sigma(\delta\tilde{x} \to -\delta\phi^c X_c\tilde{x}) + \frac{e\delta\phi^c}{2}X_c\phi.\{\partial^\sigma\phi, e^{-1}\}_\star\\ &+\frac{e\delta\phi^c}{2}.\{\partial^\sigma\phi_c, e^{-1}\}_\star + ee_b^\sigma\Big\{-\mathcal{L}_\star^\Omega\star(\delta\phi^b e^{-1}) + \delta\phi^b(\mathcal{L}_\star^\Omega\star e^{-1})\\ &+T(\Delta)\big(X_c\big(\mathcal{L}_\star^\Omega\big), \widetilde{X}^b(\delta\phi^c e^{-1})\big) + \frac{1}{2}T(\Delta)\big(\partial_\mu\big(\delta\phi^c e_c^\rho\big)\partial_\rho\phi, \widetilde{X}^b\{\partial^\mu\phi, e^{-1}\}_\star\big)\\ &+S(\Delta)\big(\partial_\mu\phi, \widetilde{X}^b\big(\big(\partial_\mu\big(\delta\phi^c e_c^\rho\big)\partial_\rho\phi\big)\star e^{-1}\big)\big)\Big\}\\ &+\frac{1}{2}ee_b^\sigma\Big\{-T(\Delta)\big(\delta\phi^c X_c\partial_\mu\phi_a, \widetilde{X}^b\{\partial^\mu\phi^a, e^{-1}\}_\star\big)\\ &-2S(\Delta)\big(\partial^\mu\phi_a, \widetilde{X}^b((\delta\phi^c X_c\partial_\mu\phi^a)\star e^{-1})\big)\\ &+2S(\Delta)\big(\partial_\mu\phi_a, \widetilde{X}^b(\partial^\mu\delta\phi^a\star e^{-1})\big) + T(\Delta)(\partial_\mu\delta\phi_a, \widetilde{X}^b\{\partial^\mu\phi^a, e^{-1}\}_\star)\Big\},\end{aligned} \tag{59}$$

where

$$\begin{aligned}\mathcal{K}^\sigma(-\delta\phi^c X_c\phi) &\equiv \mathcal{K}^\sigma(\delta\phi \to -\delta\phi^c X_c\phi) = -\frac{e\delta\phi^c X_c\phi}{2}.\{\partial^\sigma\phi, e^{-1}\}_\star\\ &-ee_b^\sigma\bigg[T(\Delta)\bigg(\partial_\mu(\delta\phi^c X_c\phi), \frac{\widetilde{X}^b}{2}\{\partial^\mu\phi, e^{-1}\}_\star\bigg)\\ &+S(\Delta)(\partial_\mu\phi, \widetilde{X}^b(\partial^\mu(\delta\phi^c X_c\phi)\star e^{-1})) + \frac{m^2}{2}T(\Delta)(\delta\phi^c X_c\phi, \widetilde{X}^b\{\phi, e^{-1}\}_\star)\\ &+m^2 S(\Delta)(\phi, \widetilde{X}^b(\delta\phi^c X_c\phi\star e^{-1})) + \frac{\lambda}{4!}T(\Delta)(\delta\phi^c X_c\phi, \widetilde{X}^b\{\phi\star\phi, \{\phi, e^{-1}\}_\star\}_\star)\\ &+\frac{\lambda}{12}S(\Delta)(\phi, \widetilde{X}^b(\delta\phi^c X_c\phi\star\phi\star\phi\star e^{-1}))\\ &+\frac{\lambda}{12}S(\Delta)(\phi\star\phi, \widetilde{X}^b(\delta\phi^c X_c\phi\star\phi\star e^{-1}))\\ &+\frac{\lambda}{12}S(\Delta)(\phi\star\phi\star\phi, \widetilde{X}^b(\delta\phi^c X_c\phi\star e^{-1}))\\ &+\frac{\Omega^2}{8}T(\Delta)(\delta\phi^c X_c\phi, \widetilde{X}^b\{\tilde{x}, \{e^{-1}, \{\tilde{x}, \phi\}_\star\}_\star\}_\star)\\ &+\frac{\Omega^2}{4}S(\Delta)(\tilde{x}, \widetilde{X}^b(\delta\phi^c X_c\phi\star\{\tilde{x}, \phi\}_\star\star e^{-1}))\\ &+\frac{\Omega^2}{4}S(\Delta)(\{\tilde{x}, \phi\star\tilde{x}\}_\star, X^b(\delta\phi^c X_c\phi\star e^{-1}))\\ &+\frac{\Omega^2}{4}S(\Delta)(\{\phi, \tilde{x}\}_\star, \widetilde{X}^b(\delta\phi^c X_c\phi\star\tilde{x}\star e^{-1}))\bigg]\end{aligned} \tag{60}$$

and

$$\begin{aligned}\mathcal{R}^\sigma(-\delta\phi^c X_c\tilde{x}) &\equiv \mathcal{R}^\sigma(\delta\tilde{x} \to -\delta\phi^c X_c\tilde{x}) = -\frac{\Omega^2}{8}ee_b^\sigma\{T(\Delta)(\delta\phi^c X_c\tilde{x}, \widetilde{X}^b\{\phi, \{e^{-1}, \{\tilde{x}, \phi\}_\star\}_\star\}_\star)\\ &+2S(\Delta)(\{\tilde{x}, \phi\}_\star, \widetilde{X}^b(\delta\phi^c X_c\tilde{x}\star\phi\star e^{-1}))\\ &+2S(\Delta)(\{\phi, \tilde{x}\star\phi\}_\star, \widetilde{X}^b(\delta\phi^c X_c\tilde{x}\star e^{-1}))\\ &+2S(\Delta)(\phi, \widetilde{X}^b(\delta\phi^c X_c\tilde{x}\star\{\tilde{x}, \phi\}_\star\star e^{-1}))\}.\end{aligned} \tag{61}$$

$\mathcal{K}^\sigma$ keeps the previous defined expression in (26). In contrast to the result in [1] for ordinary $\phi^4_\star$ theory, the twisted GW action is not invariant under global translation. Now imposing the constraint $\mathcal{E}_{\tilde{x}} = \frac{\delta \mathcal{S}^\Omega_\star}{\delta \tilde{x}} = 0$ giving

$$e\frac{\Omega^2}{8}\{\phi, \{e^{-1}, \{\tilde{x}, \phi\}_\star\}_\star\}_\star = 0 \tag{62}$$

coupled to the transformations

$$\delta\phi = -\epsilon^\nu \partial_\nu \phi \qquad \delta\phi^c = -\epsilon^\nu \partial_\nu \phi^c \qquad \epsilon^\nu = \text{constant} \tag{63}$$

that we substitute into (54) and taking into account $e^a_\nu = \partial_\nu \phi^a$, we infer from the relation

$$0 = \delta \mathcal{S}^\Omega_\star = \int_M \mathrm{d}^D x \epsilon^\nu \partial_\mu \mathcal{T}^\mu_\nu \tag{64}$$

the EMT

$$\begin{aligned}\mathcal{T}^\mu_\nu = &-\frac{e}{2}(\partial_\nu\phi)\{\partial^\mu\phi, e^{-1}\}_\star - \frac{e}{2}(\partial_\nu\phi_c)\{\partial^\mu\phi^c, e^{-1}\}_\star \\ &+ e e^\mu_b\Big\{\mathcal{L}^\Omega_\star \star (e^{-1}\partial_\nu\phi^b) + T(\Delta)(X_c\mathcal{L}^\Omega_\star, \widetilde{X}^b(e^{-1}\partial_\nu\phi^c)) \\ &+ \Omega^2\Theta^{-1}_{\gamma\nu}[S(\Delta)(\{\tilde{x}^\gamma, \phi\}_\star, \widetilde{X}^b(\phi \star e^{-1})) \\ &+ S(\Delta)(\{\phi, \tilde{x}^\gamma \star \phi\}_\star, \widetilde{X}^b(e^{-1})) + S(\Delta)(\phi, \widetilde{X}^b\{\tilde{x}^\gamma, \phi\}_\star \star e^{-1})]\Big\}.\end{aligned} \tag{65}$$

This tensor is neither symmetric nor locally conserved. In the case of a standard Moyal product, it reduces to the NC EMT computed in [13] and its regularization can be worked out in the same way as done in that work. Similarly, using the transformation

$$\delta\phi = -\epsilon^\nu\partial_\nu\phi = -\epsilon^{\nu\rho}x_\rho\partial_\nu\phi \qquad \delta\phi^c = -\epsilon^\nu\partial_\nu\phi^c = -\epsilon^{\nu\rho}x_\rho\partial_\nu\phi^c \qquad \epsilon^\nu = \epsilon^{\nu\rho}x_{\rho,} \tag{66}$$

where $\epsilon^{\nu\rho}$ is an infinitesimal constant skew symmetric Lorentz tensor, and $\epsilon^{\nu\rho}x_{[\nu}\partial_{\rho]}\phi = -2\epsilon^{\nu\rho}x_\rho\partial_\nu\phi$ substituted into (54) yields

$$0 = \delta\mathcal{S}^\Omega_\star = \int_M \mathrm{d}^D x \epsilon^{\nu\rho}\partial_\mu \mathcal{M}^\mu_{\nu\rho} \tag{67}$$

which affords the AMT as

$$\begin{aligned}\mathcal{M}^\mu_{\nu\rho} = &\frac{e}{4}x_{[\nu}\partial_{\rho]}\phi\{\partial^\mu\phi, e^{-1}\}_\star + \frac{e}{4}x_{[\nu}\partial_{\rho]}\phi_c\{\partial^\mu\phi^c, e^{-1}\}_\star - \frac{ee^\mu_b}{2}\left(\mathcal{L}^\Omega_\star \star (e^{-1}x_{[\nu}\partial_{\rho]}\phi^b)\right) \\ &+ \frac{ee^\mu_b}{2}\Big\{T(\Delta)(X_c\mathcal{L}^\Omega_\star, \widetilde{X}^b(e^{-1}x_{[\nu}\partial_{\rho]}\phi^c)) - T(\Delta)\left(\partial_{[\nu}\phi, \frac{1}{2}\widetilde{X}^b(\{\partial_{\rho]}\phi, e^{-1}\}_\star)\right) \\ &- T(\Delta)\left(\partial_{[\nu}\phi^d, \frac{1}{2}\widetilde{X}^b(\{\partial_{\rho]}\phi_d, e^{-1}\}_\star)\right) + S(\Delta)(\partial_{[\nu}\phi, \widetilde{X}^b(\partial_{\rho]}\phi \star e^{-1})) \\ &+ S(\Delta)(\partial_{[\nu}\phi_d, \widetilde{X}^b(\partial_{\rho]}\phi^d \star e^{-1})) \\ &- \frac{\Omega^2}{4}\Theta^{-1}_{\gamma[\nu}[T(\Delta)(x_{\rho]}, \widetilde{X}^b(\{\phi, \{e^{-1}, \{\tilde{x}^\gamma, \phi\}_\star\}_\star\}_\star)) \\ &+ 2S(\Delta)(\{\tilde{x}^\gamma, \phi\}_\star, \widetilde{X}^b(x_{\rho]} \star \phi \star e^{-1})) + 2S(\Delta)(\{\phi, \tilde{x}^\gamma \star \phi\}_\star, \widetilde{X}^b(x_{\rho]} \star e^{-1})) \\ &+ 2S(\Delta)(\phi, \widetilde{X}^b(x_{\rho]} \star \{\tilde{x}, \phi\}_\star \star e^{-1}))]\Big\}.\end{aligned} \tag{68}$$

This angular momentum tensor is not conserved, in contrast to the result obtained for the nonrenormalizable twisted $\phi^{\star 4}$ model studied in [1]. This analysis is compatible with the previous GW model investigation [14]. One recovers the canonical angular momentum tensor of the decoupled fields in the commutative limit. Defining now the dilatation transformation by

$$x \to x' = \epsilon x \qquad \phi(x) \to \phi'(x') = \phi'(\epsilon x) = \epsilon^{-\Delta}\phi(x), \tag{69}$$

where $\epsilon$ is a constant number, and $\Delta$ is the scale dimension of the field $\phi$, we note that the GW action is invariant over dilatation symmetry if $\Delta = 0$ and $\epsilon = \pm 1$, implying

$$x' = x \qquad \phi'(x) = \phi(x) \qquad \text{or} \qquad x' = -x \qquad \phi'(-x) = \phi(x), \tag{70}$$

which is nothing but a parity transformation of the spacetime inducing a conserved current:

$$\mathcal{D}^\mu = \mathcal{R}^\mu(\delta\tilde{x} \to -2\tilde{x}) - 2x^\mu(\mathcal{L}_\star^\Omega \star e^{-1})e. \tag{71}$$

Finally, the EMT, AMT and DC can be computed under the well-defined field values at the boundary, i.e. $\int e\, \mathrm{d}^D x\, X_b S(\Delta)(f, \widetilde{X}^b g) = 0$, to give simplified expressions. In this case, there follow

$$\mathcal{T}_\nu^\mu = -\frac{e}{2}(\partial_\nu\phi)\{\partial^\mu\phi, e^{-1}\}_\star - \frac{e}{2}(\partial_\nu\phi_c)\{\partial^\mu\phi^c, e^{-1}\}_\star \\ + e e_b^\mu\left\{\mathcal{L}_\star^\Omega \star (e^{-1}\partial_\nu\phi^b) + T(\Delta)(X_c\mathcal{L}_\star^\Omega, \widetilde{X}^b(e^{-1}\partial_\nu\phi^c))\right\} \tag{72}$$

and

$$\mathcal{M}_{\nu\rho}^\mu = \frac{e}{4}x_{[\nu}\partial_{\rho]}\phi\{\partial^\mu\phi, e^{-1}\}_\star + \frac{e}{4}x_{[\nu}\partial_{\rho]}\phi_c\{\partial^\mu\phi^c, e^{-1}\}_\star - \frac{e e_b^\mu}{2}(\mathcal{L}_\star^\Omega \star (e^{-1}x_{[\nu}\partial_{\rho]}\phi^b)) \\ + \frac{e e_b^\mu}{2}\left\{T(\Delta)(X_c\mathcal{L}_\star^\Omega, \widetilde{X}^b(e^{-1}x_{[\nu}\partial_{\rho]}\phi^c)) - T(\Delta)\left(\partial_{[\nu}\phi, \frac{1}{2}\widetilde{X}^b(\{\partial_{\rho]}\phi, e^{-1}\}_\star)\right)\right. \\ - T(\Delta)\left(\partial_{[\nu}\phi^d, \frac{1}{2}\widetilde{X}^b(\{\partial_{\rho]}\phi_d, e^{-1}\}_\star)\right) \\ \left. - \frac{\Omega^2}{4}\Theta^{-1}_{\gamma[\nu}T(\Delta)(x_{\rho]}, \widetilde{X}^b(\{\phi, \{e^{-1}, \{\tilde{x}^\gamma, \phi\}_\star\}_\star\}_\star))\right\} \tag{73}$$

and the current of dilatation symmetry expressed as

$$\mathcal{D}^\mu = -\Omega^2 e e_b^\mu T(\Delta)(\tilde{x}, \widetilde{X}^b\{\phi, \{e^{-1}, \{\tilde{x}, \phi\}_\star\}_\star\}) - 2x^\mu(\mathcal{L}_\star^\Omega \star e^{-1})e. \tag{74}$$

## 4. Concluding remarks

In this paper, we have implemented the dynamical noncommutativity introduced by Aschieri *et al* [1] in the new class of renormalizable NC field theories built on the Grosse and Wulkenhaar (GW) $\phi^{\star 4}$ scalar field model defined in Euclidean space. The corresponding equations of motion and Noether currents have been studied and explicitly computed taking into account different contributions from velocity term, mass term, $\phi^{\star 4}$ interaction and GW harmonic interaction term. When $e_\mu^a = \delta_\mu^a$ the $\star$-product between any two functions reduces to the Moyal product, as already observed in [1]. The field $\phi$ acts as a source for the noncommutativity field $\phi^c$.

Our investigation has shown that the twisted GW action is not invariant under global translation. Such an undesirable feature has been got round by imposing a constraint on the Lagrangian action, which is nothing but the equation of motion governing the GW harmonic term. The previous works [1, 6] have pointed out that the ordinary $\phi^4$-theory leads to nonlocally conserved and symmetric EMT and AMT while the twisted nonrenormalizable $\phi^4$-theory restores the local conservation of these tensors. Contrarily, both ordinary GW [13, 14] and twisted GW models provide nonlocally conserved and nonsymmetric EMT, AMT and DC due to the presence of the harmonic term Ω. Fortunately, as shown in [13], all these physical quantities can be subjected to well-known Jackiw and Wilson regularization procedures to acquire the local conservation property.

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