J. Math. Anal. Appl. 422 (2015) 1–26

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Weighted eigenvalue problems for quasilinear elliptic operators with mixed Robin–Dirichlet boundary conditions

Mabel Cuesta [a,*], Liamidi Leadi [b]

[a] Université du Littoral ULCO, LMPA, 50 rue F. Buisson, 62228, Calais, France
[b] Institut de Mathématiques et de Sciences Physiques, Université d'Abomey Calavi, 613 Porto-Novo, République du Bénin

ARTICLE INFO

Article history:
Received 5 May 2014
Available online 19 August 2014
Submitted by M. del Pino

Keywords:
p-Laplacian
Indefinite weights
Indefinite potential
Principal eigenvalues Robin boundary conditions

ABSTRACT

We investigate the existence of principal eigenvalues type problems with weights for the quasilinear operator $-\Delta_p+V\psi_p$ with mixed weighted Robin–Dirichlet boundary conditions in a bounded regular domain. We also give some results on the existence of nonprincipal eigenvalues.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a smooth boundary $\partial\Omega$ and let ν be its outer normal defined everywhere. Let V be a bounded function defined in Ω and σ a smooth function defined on $\partial\Omega$. It was pointed out in [9,8] that the Fourier analysis for parabolic problems with dynamic boundary conditions leads, through a separation of variables, to the following eigenvalue problem with Robin type boundary conditions

$$\begin{cases} -\Delta u + V(x)u = \lambda u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \lambda\sigma(x)u & \text{on } \partial\Omega. \end{cases} \tag{L}$$

The complete analysis of this eigenvalue problem when $V \geq 0$ has been done in [9] in the case $\sigma = cst$, and in [8] in the case $\sigma \neq cst$ (for a slighter general operator than the laplacian). As a common feature, it appears that problem (L) possesses an infinite sequence of positive eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$ if σ^+ (the positive part of σ) is $\not\equiv 0$, and an infinite sequence of negative eigenvalues $\{\lambda_{-n}\}_{n\in\mathbb{N}}$ if $\sigma^- \not\equiv 0$ and $N \geq 2$. Moreover,

* Corresponding author.
E-mail addresses: cuesta@lmpa.univ-littoral.fr (M. Cuesta), leadiare@imsp-uac.org (L. Leadi).

http://dx.doi.org/10.1016/j.jmaa.2014.08.015

$\lambda_{\pm 1}$ are both principal eigenvalue (i.e., an eigenvalue whose eigenfunctions are sign-constant) and simple (i.e. the associated eigenfunctions are each a constant multiple of one another).

It is also well known that the spectra of the $-\Delta + V$, in the case $V^- \not\equiv 0$, could present features different from those of the spectra in the case $V \geq 0$. Problem (L) with V indefinite and Dirichlet boundary conditions has been extensively studied for instance by Allegretto–Mingarelli [4], Fleckinger–Hernandez–de Thelin [15] and Lopez-Gomez [20] among others.

Our intention in this paper is to initiate the *study of the spectrum* of the more general problem

$$\begin{cases} -\Delta_p u + V(x)|u|^{p-2}u = \lambda m(x)|u|^{p-2}u & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda \sigma |u|^{p-2}u & \text{on } \Gamma_1, \\ u = 0 & \text{on } \Gamma_2, \end{cases} \tag{P}$$

with V, m and σ indefinite. Here $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$, denotes the p-Laplacian operator for $p > 1$. We will assume that Ω is a bounded smooth domain and that $\partial\Omega$ splits up in two sets Γ_1 and Γ_2 which are connected and closed $(n-1)$-manifolds.

The existence of principal eigenvalues for the quasilinear equation in problem (P) with Dirichlet boundary condition and V, m indefinite has been treated by Binding–Huang [10,11] and [13]. It appears that sometimes there are not principal eigenvalues, a phenomenon that depends, loosely speaking, on how big the negative part of V with respect to the negative part of m is.

Problem (P) for $p = 2$ with $V \equiv 0$, $\Gamma_2 = \emptyset$ and m indefinite has already been considered by Afrouzi–Brown [2] when $\sigma = cst$, and later by K. Umezu [23] for indefinite σ. This last author proved that, besides the trivial eigenvalue $\lambda = 0$, problem (P) possesses a *unique* positive principal eigenvalue if and only if

$$\oint_{\partial\Omega} \sigma \, d\rho + \int_{\Omega} m \, dx < 0.$$

Here $d\rho$ stands for the surface element of $\partial\Omega$.

In the case $V \not\equiv 0$ and possibly indefinite, the situation is much different since the energy functional $E_V(u) \stackrel{\text{def}}{=} \int_\Omega (|\nabla u|^p + V|u|^p)dx$ is indefinite. One approach to find principal eigenvalues that has been used by many authors is to define a new eigenvalue problem for each fixed λ and to construct "an eigenvalue curve" as λ varies. We apply this approach in Section 3 and we give in Section 4 a necessary and sufficient condition for the existence of eigenvalues in terms of the infimum of E_V over the set of functions $\mathcal{G}$ satisfying $\int_\Omega |u|^p \, dx = 1$ and

$$\int_{\Omega} m|u|^p \, dx + \oint_{\Gamma_1} |u|^p \sigma \, d\rho = 0.$$

Indeed, this set $\mathcal{G}$ was already considered in [10,11] and [13] for the quasilinear equation of problem (P) with Dirichlet, Newmann or mixed boundary conditions. In fact, one cannot exclude that some eigenfunctions belong to $\mathcal{G}$ and, in that case, the corresponding energy levels are called in [11] "ghost states". Ghost states are interesting because they have the property of *losing of compactness*, see the discussion in Section 8 and Remark 8.6.

This paper is organized as follows. We construct the eigencurve associated to problem (P) in Section 3. The existence of principal eigenvalues is studied in Section 4. A sufficient condition for the existence of principal eigenvalues is presented in Section 5, where we also discuss the necessity of such condition for the coerciveness of the related functional. In Section 6 we prove isolation and simplicity of principal eigenvalues of problem (P). In Section 7 we investigate the coerciveness of the restricted functional and in Section 8 we prove the existence of two unbounded sequences of eigenvalues in the case where either $m^\pm \not\equiv 0$ or $\sigma^\pm \not\equiv 0$.

We also exhibit an example of problem (P) in dimension 1, with a changing-sign σ, that fails to have a double sequence of eigenvalues.

Notice that the Dirichlet–Newmann eigenvalue problem associated to (P), that is

$$\begin{cases} -\Delta_p u + V(x)|u|^{p-2}u = \lambda m(x)|u|^{p-2}u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_1, \\ u = 0 & \text{on } \Gamma_2, \end{cases}$$

corresponds to the case $\sigma \equiv 0$. As a by-product of our main result in Theorem 4.1 we will show that there exists always a unique principal eigenvalue for this problem.

2. Main assumptions and useful inequalities

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class $C^{2,\alpha}$ for some $0 < \alpha < 1$. The Lebesgue measure of a measurable set A of $\mathbb{R}^N$ will be denoted by $\lambda_N(A)$. We denote by ρ the restriction to $\partial\Omega$ of the $(N-1)$-Hausdorff measure, which coincides with the usual Lebesgue surface measure as $\partial\Omega$ is regular enough. We denote by $\nu = \nu(x)$ its outer normal at x, defined for all $x \in \partial\Omega$. We will assume that $\partial\Omega$ splits up in two sets Γ_1 and Γ_2 which are connected and closed $(n-1)$-manifolds. We allow Γ_2 to be the empty set.

Throughout this paper we always assume that the weights $V, m \in L^\infty(\Omega)$ and $\sigma \in C^{0,r}(\Gamma_1)$ for some $0 < r < 1$, are possibly indefinite. We will always assume that either m or σ is not equivalent to 0. We will denote

$$\Omega^+ \stackrel{\text{def}}{=} \{x \in \Omega \mid m(x) > 0\}, \qquad \Omega^- \stackrel{\text{def}}{=} \{x \in \Omega \mid m(x) < 0\}, \qquad \Omega^0 \stackrel{\text{def}}{=} \{x \in \Omega \mid m(x) = 0\}.$$

Similarly, we will denote Γ_1^+, Γ_1^- and Γ_1^0 the positive, negative and null set of the weight σ in Γ_1.

We denote $W^{1,p}(\Omega)$ the classical Sobolev space endowed with the classical norm

$$\|u\| := \left(\int_\Omega |\nabla u|^p \, dx + \int_\Omega |u|^p \, dx \right)^{1/p}.$$

The Lebesgue norm of $L^q(\Omega)$ will be denoted by $\|\cdot\|_q$, and the Lebesgue norm of $L^q(\partial\Omega, \rho)$ by $\|\cdot\|_{q,\partial\Omega}$, for any $q \in [1, +\infty]$.

The conjugate of any $r \in [1, +\infty]$ will be denoted by r', the critical Sobolev exponent for the embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ will be denoted by $p^* := \frac{pN}{N-p}$ if $1 < p < N$, $p^* = +\infty$ otherwise.

The *trace* operator will be denoted by γ, that is

$$\gamma : W^{1,p}(\Omega) \to W^{1-\frac{1}{p},p}(\partial\Omega, \rho).$$

Recall that there is a continuous boundary trace embedding $W^{1,p}(\Omega) \to L^q(\partial\Omega, \rho)$ for every $q \in [1, p_*]$ and that those embeddings are compact for $q \in [1, p_*[$. Here we denote by $p_* = \frac{p(N-1)}{(N-p)^+}$ the critical exponent for the above trace embedding. For the properties of γ (especially the surjectivity) we refer to [1].

We put $W \stackrel{\text{def}}{=} \{u \in W^{1,p}(\Omega) \mid \gamma(u) = 0 \text{ on } \Gamma_2\}$. One can show that

$$\|u\|_W := \begin{cases} \int_\Omega |\nabla u|^p \, dx + \int_{\partial\Omega} |u|^p d\rho & \text{if } \Gamma_2 = \emptyset; \\ \int_\Omega |\nabla u|^p \, dx & \text{if } \Gamma_2 \neq \emptyset, \end{cases}$$

is a norm on W equivalent to $\|\cdot\|$.

Let us recall Picone's identity [3]:

Lemma 2.1. *(See [3].) Let $w \geq 0$, $v > 0$ be two continuous functions in Ω differentiable a.e. Denote*

$$L(w,v) = |\nabla w|^p + (p-1)\frac{w^p}{v^p}|\nabla v|^p - p\frac{w^{p-1}}{v^{p-1}}|\nabla v|^{p-2}\nabla v \nabla w,$$
$$R(w,v) = |\nabla w|^p - |\nabla v|^{p-2}\nabla\left(\frac{w^p}{v^{p-1}}\right)\nabla v.$$

Then **(i)** $L(w,v) = R(w,v)$, **(ii)** $L(w,v) \geq 0$ *a.e. and* **(iii)** *Assume that $\frac{w}{v} \in W^{1,1}_{loc}(\Omega)$. Then $L(w,v) = 0$ a.e. in Ω if and only if $w = kv$ for some $k \in \mathbb{R}$.*

The following property can be found for instance in [19]:

Lemma 2.2. *(See [19].) The operator $-\Delta_p : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ satisfies the so called S^+ property: for all sequence $u_n \in W^{1,p}(\Omega)$ such that $u_n \hookrightarrow u_0$ weakly in $W^{1,p}(\Omega)$ and*

$$\lim_{n\to+\infty} \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla(u_n - u_0)dx = 0,$$

it holds $\|\nabla u_n - \nabla u_0\|_p \to 0$.

3. An eigenvalue curve associated to problem (*P*)

It is well established that, in order to prove the existence of principal eigenvalues of (*P*), one fixes λ and embeds the problem into the new *eigenvalue problem of parameter* μ:

$$\begin{cases} -\Delta_p u + V(x)|u|^{p-2}u = \lambda m(x)|u|^{p-2}u + \mu|u|^{p-2}u & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda\sigma|u|^{p-2}u & \text{on } \Gamma_1, \\ u = 0 & \text{on } \Gamma_2. \end{cases} \tag{3.1}$$

A value $\mu \in \mathbb{R}$ is called an *eigenvalue* for problem (3.1) if and only if there exists $u \in W$, $u \not\equiv 0$, satisfying Eq. (*P*) in the weak sense, i.e., $\forall w \in W$

$$\int_\Omega \left(|\nabla u|^{p-2}\nabla u \cdot \nabla w + (V - \lambda m)|u|^{p-2}uw\right)dx = \mu \int_\Omega |u|^{p-2}uw\,dx + \lambda \oint_{\Gamma_1} \sigma|u|^{p-2}uw\,d\rho.$$

The function $u \in W$ is called an *eigenfunction*. An eigenvalue is called *principal* if it possesses an eigenfunction $u > 0$ a.e. in Ω.

Remark 3.1. We recall that, by the regularity results of [14] and [18, Theorem 2], bounded weak solutions of (3.1) are of class $C^{1,\beta}(\overline{\Omega})$ for some $0 < \beta < 1$. We use here that Ω is of class $C^{2,\alpha}$ as well as that $\sigma \in C^{0,r}(\partial\Omega)$ for some $0 < \alpha$, $r < 1$. It is well known in the case $m = 0$ that any weak solution of (3.1) belongs to $L^\infty(\Omega) \cap L^\infty(\partial\Omega, \rho)$ (see for instance [17]). See Theorem A.1 in Appendix A for a more general result.

We are going to consider the smaller eigenvalue $\mu \in \mathbb{R}$ of problem (3.1). In order to do so, we define the energy functional

$$J_\lambda : W \to \mathbb{R}; \qquad J_\lambda(u) \stackrel{\text{def}}{=} E_V(u) - \lambda I(u),$$

where

$$E_V(u) \stackrel{\text{def}}{=} \int_\Omega \left(|\nabla u|^p + V|u|^p\right) dx,$$

and

$$I(u) \stackrel{\text{def}}{=} \int_\Omega m|u|^p\, dx + \oint_{\Gamma_1} \sigma|u|^p\, d\rho.$$

Let us also consider the manifold

$$S \stackrel{\text{def}}{=} \left\{ u \in W \;\Big|\; \int_\Omega |u|^p\, dx = 1 \right\}.$$

Proposition 3.2. *The value*

$$\mu_1(\lambda) \stackrel{\text{def}}{=} \inf\left\{ J_\lambda(u) \;\big|\; u \in S \right\} \in \mathbb{R}$$

is the smaller eigenvalue of (3.1). Moreover $\mu_1(\lambda)$ is principal, simple and it is the unique principal eigenvalue associated to (3.1).

If we denote by $\varphi_\lambda \in S$ the (unique) positive eigenfunction for problem (3.1), it holds that $\varphi_\lambda \in C^{1,\beta}(\overline{\Omega})$, $\varphi_\lambda > 0$ in $\Omega \cup \Gamma_1$.

Proof. The proof is quite standard but we include it for the sake of completeness. One uses the following simple estimate that can be proved by arguing by contradiction:

$\forall \epsilon > 0$ $\exists c(\epsilon) > 0$ *such that*

$$\oint_{\partial\Omega} |u|^p\, d\rho \le \epsilon \int_\Omega |\nabla u|^p\, dx + c(\epsilon) \int_\Omega |u|^p\, dx \quad \forall u \in W^{1,p}(\Omega). \tag{3.2}$$

Thus we have, for all $u \in S$ and all $\lambda \in \mathbb{R}$,

$$J_\lambda(u) \ge \left(1 - |\lambda|\epsilon\|\sigma\|_{\infty,\Gamma_1}\right) \int_\Omega |\nabla u|^p\, dx - \|V\|_\infty - |\lambda|\left(\|m\|_\infty + c(\epsilon)\|\sigma\|_{\infty,\Gamma_1}\right). \tag{3.3}$$

Then, by choosing any $0 < \epsilon < (|\lambda|\|\sigma\|_{\infty,\Gamma_1})^{-1}$ in (3.3), it comes that J_λ is bounded from below on S. Since J_λ is sequentially weakly lower semi-continuous, it follows that $\mu_1(\lambda)$ is achieved at some $u \in S$. By Lagrange multipliers rule, one concludes that $\mu_1(\lambda)$ is an eigenvalue for (3.1) and u is an associated eigenfunction. Since $J_\lambda(u) = J_\lambda(|u|)$ it follows that $|u|$ is also an eigenfunction for $\mu_1(\lambda)$. By the regularity results already quoted in Remark 3.1, $u \in C^{1,\beta}(\overline{\Omega})$ and, by the well known Strong Maximum Principle of [24], we conclude that $|u| > 0$ in Ω, so either $u > 0$ in Ω or $u < 0$ in Ω, that is, $\mu_1(\lambda)$ is principal. Moreover $u > 0$ on Γ_1 otherwise, by the Hopf maximum principle of [24], one will infer that $\frac{\partial u}{\partial \nu} < 0$ on Γ_1, in contradiction with the boundary condition of (3.1).

To prove that $\mu_1(\lambda)$ is simple, assume that $w, v \in W$ are two different eigenfunctions of (3.1) for μ_1. Thus, we can then assume that $w > 0$ and $v > 0$ in Ω. Next, one uses Picone's identity as follows. Choose for any $\eta > 0$ the function $\frac{w^p}{(v+\eta)^{p-1}}$ as a test function in Eq. (3.1) satisfied by v and w as a test function in Eq. (3.1) satisfied by w to get

$$0 \le \int_\Omega R(w, v+\eta)\, dx = \int_\Omega |\nabla w|^p\, dx - \int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla\left(\frac{w^p}{(v+\eta)^{p-1}}\right) dx$$
$$= \int_\Omega |\nabla w|^p\, dx - \int_\Omega \big(V - \lambda m - \mu_1(\lambda)\big) w^p \frac{v^p}{(v+\eta)^{p-1}}\, dx - \lambda \oint_{\Gamma_1} \sigma w^p \frac{v^p}{(v+\eta)^{p-1}}\, d\rho.$$

By letting $\eta \to 0$ it comes that $R(u, v) = 0$ and the conclusion follows. □

The curve $\mu_1 : \mathbb{R} \to \mathbb{R}$ is known as *eigencurve* associated to problem (P). This notion was first used in [10,11], for Newmann or Dirichlet boundary problems, as well as the following properties that we will prove here for the weighted Robin–Dirichlet boundary conditions.

Proposition 3.3. *Assume that m or σ are $\not\equiv 0$. The following properties hold:*

1. *μ_1 is concave, differentiable and*

$$\mu_1'(\lambda) = -\left(\int_\Omega m\varphi_\lambda^p\, dx + \oint_{\Gamma_1} \sigma\varphi_\lambda^p\, d\rho\right). \tag{3.4}$$

2. *If $m^+ \not\equiv 0$ in Ω or $\sigma^+ \not\equiv 0$ on Γ_1 then $\lim_{\lambda\to+\infty} \mu_1(\lambda) = -\infty$.*
3. *If $m^- \not\equiv 0$ in Ω or $\sigma^- \not\equiv 0$ on Γ_1 then $\lim_{\lambda\to-\infty} \mu_1(\lambda) = -\infty$.*
4. *If $m \ge 0$ in Ω and $\sigma \ge 0$ on Γ_1 (resp. $m \le 0$ in Ω and $\sigma \le 0$ on Γ_1) then μ_1 is strictly decreasing (resp. strictly increasing).*
5. *Let us denote*

$$\alpha \stackrel{\text{def}}{=} \inf\left\{E_V(u) \;\middle|\; u \in S, \int_\Omega m|u|^p\, dx + \oint_{\Gamma_1} \sigma|u|^p\, d\rho = 0\right\}. \tag{3.5}$$

Then $\alpha = \sup_{\lambda\in\mathbb{R}} \mu_1(\lambda)$.

Proof. (1) For a fixed $u \in W$ the mapping $\lambda \mapsto J_\lambda(u)$ is concave and then the infimum over S, that is $\mu_1(\lambda)$, is also concave and therefore continuous. Now let $\lambda_n \to \lambda$ and φ_n, φ_λ be the L^p-normalized positive eigenfunctions related to λ_n, λ respectively. If we apply (3.3) with $\lambda = \lambda_n$ and $u = \varphi_n$ we have, after choosing ϵ small enough, that

$$\|\nabla\varphi_n\|_p^p \le C_1$$

for some $C_1 > 0$. So we conclude that the sequence φ_n is bounded in W. Hence there exists φ_0 such that, up to a subsequence, $\varphi_n \rightharpoonup \varphi_0$ in W, strongly in $L^p(\Omega)$ and in $L^p(\partial\Omega, \rho)$. Then $\|\varphi_0\|_p = 1$ and, from

$$\mu(\lambda) = \lim_{n\to+\infty} \mu(\lambda_n) \ge E_{V-\lambda m}(\varphi_0) - \lambda \oint_{\Gamma_1} \sigma|\varphi_0|^p\, d\rho \ge \mu(\lambda),$$

we infer that $\varphi_0 = \varphi_\lambda$. Furthermore

$$\mu(\lambda_n) = E_{V-\lambda_n m}(\varphi_n) - \lambda_n \oint_{\Gamma_1} \sigma|\varphi_n|^p\, d\rho$$

$$= E_{V-\lambda m}(\varphi_n) + (\lambda - \lambda_n) \int_\Omega m|\varphi_n|^p\, dx - \lambda_n \oint_{\Gamma_1} \sigma|\varphi_n|^p\, d\rho$$

$$\geq \mu(\lambda) + (\lambda - \lambda_n) \int_\Omega m|\varphi_n|^p dx + (\lambda - \lambda_n) \oint_{\Gamma_1} \sigma|\varphi_n|^p\, d\rho$$

and replacing λ (resp. φ) by λ_n (resp. φ_n) in this inequality we have, for $\lambda_n > \lambda$,

$$-\int_\Omega m|\varphi_n|^p\, dx - \oint_{\Gamma_1} \sigma|\varphi_n|^p\, d\rho \leq \frac{\mu(\lambda_n) - \mu(\lambda)}{\lambda_n - \lambda} \leq -\int_\Omega m|\varphi_\lambda|^p\, dx - \oint_{\Gamma_1} \sigma|\varphi_\lambda|^p\, d\rho.$$

Passing to the limit we get (3.4). A similar argument holds if $\lambda_n < \lambda$.

(2) If $m^+ \not\equiv 0$ then it is easy to see that there exists $u_0 \in W_0^{1,p}(\Omega) \cap S$ such that $\int_\Omega m|u_0|^p\, dx > 0$. Indeed, one can choose u_0 as the regularization of the characteristic function of any small ball B strictly contained in Ω for which $m^+ \not\equiv 0$ in B. Hence

$$\mu_1(\lambda) \leq E_V(u_0) - \lambda \int_\Omega m|u_0|^p\, dx \to -\infty \quad \text{as } \lambda \to +\infty.$$

If $m^+ \equiv 0$ in Ω and $\sigma^+ \not\equiv 0$ on Γ_1 we use Lemma 3.4 below to get the existence of $u_0 \in S$ such that $\int_\Omega m|u_0|^p\, dx + \oint_{\Gamma_1} \sigma|u_0|^p\, d\rho = 1$ and the conclusion follows.

(3) Similar to the previous case.

(4) If $m \geq 0$ in Ω and $\sigma \geq 0$ on Γ_1, the result is clear from the fact that

$$\int_\Omega m|\varphi_\lambda|^p\, dx + \oint_{\Gamma_1} \sigma|\varphi_\lambda|^p\, d\rho > 0$$

for any $\lambda \in \mathbb{R}$. Indeed, if $\lambda_1 < \lambda_2$ then

$$\mu(\lambda_1) = E_V(\varphi_{\lambda_1}) - \lambda_1\Big(\int_\Omega m|\varphi_{\lambda_1}|^p\, dx + \oint_{\Gamma_1} \sigma|\varphi_{\lambda_1}|^p\, d\rho\Big)$$

$$> E_V(\varphi_{\lambda_1}) - \lambda_2\Big(\int_\Omega m|\varphi_{\lambda_1}|^p\, dx + \oint_{\Gamma_1} \sigma|\varphi_{\lambda_1}|^p\, d\rho\Big) \geq \mu(\lambda_2).$$

(5) Let us prove that $\sup_{\lambda\in\mathbb{R}} \mu(\lambda) = \alpha$. Notice that trivially

$$\alpha \geq \mu(\lambda)$$

for all $\lambda \in \mathbb{R}$. We distinguish the following two complementary cases:

(a1) Either $m \geq 0$ and $\sigma \geq 0$ or $m \leq 0$ and $\sigma \leq 0$. In this first alternative we know that $\mu(\lambda)$ is strictly decreasing so then

$$\sup \mu(\lambda) = \lim_{\lambda\to-\infty} \mu(\lambda). \tag{3.6}$$

Let $\lambda_n \to -\infty$ when $n \to \infty$ and let φ_n be the associated L^p-normalized eigenfunction. We have

$$\mu(\lambda_n) = E_{V-\lambda_n m}(\varphi_n) - \lambda_n \oint_{\Gamma_1} \sigma|\varphi_n|^p\, d\rho \geq E_V(\varphi_n) \geq \|\varphi_n\|^p - 1 - \|V\|_\infty$$

for all $\lambda_n \leq 0$. Thus the sequence φ_n is bounded in W so there exists $\varphi \in W$ such that, up to a subsequence, $\varphi_n \rightharpoonup \varphi$ in W, strongly in $L^p(\Omega) \cap L^p(\partial\Omega, \rho)$. Thus $\|\varphi\|_p = 1$ and

$$\alpha \geq \lim_{n\to\infty} \mu(\lambda_n) \geq E_V(\varphi) - \lim_{n\to\infty} \lambda_n \Big(\int_\Omega m|\varphi_n|^p \, dx + \oint_{\Gamma_1} \sigma |\varphi_n|^p \, d\rho \Big). \tag{3.7}$$

If $m > 0$ a.e. on Ω then, from (3.7), we have

$$\lim_{\lambda\to-\infty} \mu(\lambda) = +\infty = \alpha.$$

If $\lambda_N(\{x \in \Omega \mid m(x) = 0\}) > 0$ then from Proposition 3.5(iii) $\alpha < +\infty$ and therefore μ is bounded from above. We conclude from (3.7) that $\int_\Omega m|\varphi|^p \, dx + \oint_{\Gamma_1} \sigma|\varphi|^p \, d\rho = 0$ and then φ is admissible in the definition of α. Again from (3.7), we get

$$\alpha \geq \lim_{n\to\infty} \mu(\lambda_n) \geq E_V(\varphi) \geq \alpha$$

and the result follows from (3.6). If $m \leq 0$ and $\sigma \leq 0$ we argue similarly.

(a2) Either $m^+ \not\equiv 0$ and $\sigma^- \not\equiv 0$ or $m^- \not\equiv 0$ and $\sigma^+ \not\equiv 0$. In both cases $\alpha < +\infty$ from Proposition 3.5 below and it follows from (1)–(3) that μ is bounded from above. Then $\sup_{\lambda\in\mathbb{R}} \mu(\lambda)$ is achieved at some λ_0 that satisfies $0 = \mu'(\lambda_0) = -\int_\Omega m\varphi_{\lambda_0}^p \, dx - \oint_{\Gamma_1} \sigma\varphi_{\lambda_0}^p \, d\rho$. We conclude that φ_{λ_0} is admissible in the definition of α and hence

$$\alpha \leq E_V(\varphi_{\lambda_0}) = \mu(\lambda_0) = \sup_{\lambda\in\mathbb{R}} \mu(\lambda). \qquad \square$$

Lemma 3.4. *If $\sigma^- \not\equiv 0$ or $m^- \not\equiv 0$ then there exists $u_0 \in W$ such that $supp\, u_0 \subset \Omega^-$, $supp\, \gamma(u_0) \subset \Gamma_1^-$ and*

$$\int_\Omega m|u_0|^p \, dx + \oint_{\Gamma_1} \sigma|u_0|^p d\rho < 0. \tag{3.8}$$

Proof. If $m^- \not\equiv 0$ one can take $u_0 \in W_0^{1,p}(\Omega)$ such that $\int_\Omega m|u_0|^p \, dx < 0$ and $supp\, u_0 \subset \Omega^-$. If $m^- \equiv 0$ pick any $0 < \psi \in C^\infty(\Gamma_1)$ with $supp\, \psi \subset \Gamma_1^-$. Put $a := \oint_{\Gamma_1} \sigma\psi^p \, d\rho < 0$ and take a function $0 \leq v \in W^{1,\infty}(\Omega)$ such that $\gamma(v) = \psi$. Consider for $\epsilon > 0$ small a cut-function $\xi_\epsilon \in C^\infty(\mathbb{R}^N)$ such that

$$\xi_\epsilon \equiv 1 \text{ in } \Omega_\epsilon, \qquad \xi_\epsilon \equiv 0 \text{ in } \mathbb{R}^N \setminus \Omega_{\epsilon/2},$$

where

$$\Omega_\epsilon := \big\{ x \in \Omega \mid dist(x, \Gamma_1) > \epsilon \big\}.$$

Thus $u_0 := (1 - \xi_\epsilon)v \in W$ and $\gamma(u_0) = \psi$. Choose $\epsilon > 0$ small enough to have

$$\lambda_N(\Omega \setminus \Omega_\epsilon) < \frac{-a}{2\|m\|_\infty \|v\|_\infty^p}.$$

Hence

$$\int_\Omega |m||u_0|^p \, dx \leq \|m\|_\infty \|v\|_\infty^p \lambda_N(\Omega \setminus \Omega_\epsilon) \leq -\frac{1}{2} \oint_{\Gamma_1} \sigma|u_0|^p \, d\rho$$

and therefore

$$\int_{\Omega} m|u_0|^p\,dx + \oint_{\Gamma_1} \sigma|u_0|^p d\rho < 0. \qquad \square$$

Let us state in which cases the value α is finite.

Proposition 3.5. *Let us define the set $\mathcal{G}$ as*

$$\mathcal{G} \stackrel{\text{def}}{=} \left\{ u \in S \;\middle|\; \int_{\Omega} m|u|^p\,dx + \oint_{\Gamma_1} \sigma|u|^p\,d\rho = 0 \right\}. \tag{3.9}$$

Then $\mathcal{G} \neq \emptyset$ if and only if either (i) $m^{\pm} \not\equiv 0$ or (ii) $m^+ \not\equiv 0$ and $\sigma^- \not\equiv 0$ or $m^- \not\equiv 0$ and $\sigma^+ \not\equiv 0$ or (iii) $\lambda_N(\{x \in \Omega \mid m(x) = 0\}) > 0$.

In all these cases the value α defined in (3.5) is achieved at some $0 \leq \xi_0 \in W$.

Proof. (i) If $m^{\pm} \not\equiv 0$ one can always find function on $W_0^{1,p}(\Omega)$ belonging to $\mathcal{G}$ by taking, for instance, $u = u_1 - u_2$ with u_1 a positive function with support on Ω^+, u_2 a positive function with support on Ω^- and, after rescaling, then make $u \in \mathcal{G}$.

(ii) If, say, $m^+ \not\equiv 0$ and $\sigma^+ \not\equiv 0$, we pick any $0 < \psi \in C^{\infty}(\Gamma_1)$ such that $a := \oint_{\Gamma_1} \sigma\psi^p\,d\rho < 0$. Let $v \in W^{1,\infty}(\Omega)$ such that $\gamma(v) = \psi$ and $u_0 = (1 - \xi_\epsilon)v$ with ξ_ϵ, as in Lemma 3.4 above. We now choose $\epsilon > 0$ small enough to guarantee that $\lambda_N(\Omega_\epsilon \cap \Omega^+) \neq 0$. Let us take another function $0 \leq w \in W_0^{1,p}(\Omega_\epsilon)$ such that

$$\int_{\Omega_\epsilon} m w^p\,dx = -a.$$

Hence, as u_0 and w have disjoint supports,

$$\int_{\Omega} m|w + u_0|^p dx + \oint_{\Gamma_1} \sigma|u_0 + w|^p d\rho = 0.$$

(iii) If $\lambda_N(\Omega^0) > 0$ we can always find a function $0 \not\equiv u \in W_0^{1,p}(\Omega)$ satisfying $\int_{\Omega} m|u|^p\,dx = 0$, so, after normalization, u is admissible in the definition of α.

Finally notice that any minimizing sequence of α is bounded by the estimate

$$E_V(u) \geq \int_{\Omega} |\nabla u|^p\,dx - \|V\|_{\infty}$$

valid for any $u \in S$. Then, by standard arguments, one can show that α is achieved at some $\xi_0 \geq 0$. $\square$

4. Existence of principal eigenvalues of (*P*)

As a consequence of Proposition 3.3 we prove our main result of this section. Recall that the value α has been defined in (3.5) and the set $\mathcal{G}$ in (3.9).

Theorem 4.1.

1. (a) *If $m \geq 0$ in Ω and $\sigma \geq 0$ on Γ_1 then there exists a principal eigenvalue of (P) if and only if $\alpha > 0$. The principal eigenvalue is unique and will be denoted by λ_1. It is characterized by*

$$\lambda_1 = \min_{\mathcal{M}^+} E_V, \tag{4.1}$$

where $\mathcal{M}^+ := \{u \in W \mid I(u) = 1\} \neq \emptyset$.

(b) *If* $m \leq 0$ *in* Ω *and* $\sigma \leq 0$ *on* Γ_1 *then there exists a principal eigenvalue of* (P) *if and only if* $\alpha > 0$. *The principal eigenvalue is unique and will be denoted by* λ_{-1}. *It is characterized by*

$$\lambda_{-1} = -\min_{\mathcal{M}^-} E_V, \tag{4.2}$$

where $\mathcal{M}^- := \{u \in W \mid I(u) = -1\} \neq \emptyset$.

2. *If either* m *and* σ *are definite but with opposite sign or one of them is indefinite then there exists a principal eigenvalue of* (P) *if and only if* $\alpha \geq 0$. *More precisely:*
   (a) *if* $\alpha > 0$ *then* (P) *admits exactly two principal eigenvalues* $\lambda_{-1} < \lambda_1$, *with* λ_1 *characterized as in* (4.1) *and* λ_{-1} *characterized as in* (4.2).
   (b) *If* $\alpha = 0$ *then* (P) *has a unique principal eigenvalue* λ_* *given by*

$$\lambda_* = \inf_{\mathcal{M}^+} E_V = -\inf_{\mathcal{M}^-} E_V.$$

These infima are not achieved. Moreover a function $u \in \mathcal{G}$ *satisfies* $E_V(u) = \alpha$ *if and only if* $u \in S$ *is an eigenfunction associated to* λ_*.

Whenever they exist, one has $\lambda_{-1} \leq \lambda_1$.

By the regularity results mentioned earlier, the eigenfunctions associated to principal eigenvalues belong to $C^{1,\beta}(\overline{\Omega})$ and are > 0 in $\Omega \cup \Gamma_1$.

Proof. We only prove (2)(b) since the proof of the other statements can be carried with minor changes from the proof of [13, Theorem 7]. If $\alpha = 0$ then $\mu(\lambda_0) = 0$ at some λ_0, providing a principal eigenvalue of (P). By Lemma 7.1 below this point λ_0 is unique. Moreover $\mu'(\lambda_0) = -\int_\Omega m|\varphi_{\lambda_0}|^p\, dx - \oint_{\Gamma_1} \sigma|\varphi_{\lambda_0}|^p\, d\rho = 0$. Let us prove that $\lambda_0 = \inf_{\mathcal{M}^+} E_V$. Take $u \in \mathcal{M}^+$ and assume that $u \geq 0$ by replacing u by $|u|$ if necessary. Picone's identity (cf. Lemma 2.1) applied to $u_T \overset{\text{def}}{=} \min\{u, T\}$ and φ_{λ_0} yields (notice that $\frac{u_T^p}{\varphi_{\lambda_0}^{p-1}} \in W^{1,1}_{loc}(\Omega)$)

$$\begin{aligned}
0 &\leq \int_\Omega L(u_T, \varphi_{\lambda_0})\, dx = \int_\Omega R(u_T, \varphi_{\lambda_0})\, dx \\
&= \int_\Omega \left[|\nabla u_T|^p - |\nabla \varphi_{\lambda_0}|^{p-2} \nabla\left(\frac{u_T^p}{\varphi_{\lambda_0}^{p-1}}\right) \nabla \varphi_{\lambda_0} \right] dx \\
&= \int_\Omega |\nabla u_T|^p\, dx + \int_\Omega V u_T^p\, dx \\
&\quad - \lambda_0 \int_\Omega m u_T^p\, dx - \lambda_0 \oint_{\Gamma_1} \sigma u_T^p\, d\rho
\end{aligned}$$

Now we let $T \to \infty$ to get $E_V(u) \geq \lambda_0$. Consider the sequence $u_n \in \mathcal{M}^+$ given by

$$u_n = \frac{\varphi_{\lambda_0} + \frac{\psi}{n}}{I(\varphi_{\lambda_0} + \frac{\psi}{n})^{\frac{1}{p}}}, \tag{4.3}$$

for some fixed $0 \leq \psi \in C^\infty(\overline{\Omega})$, $\psi = 0$ on Γ_2, such that $I(\psi) > 0$, and $\langle I'(\varphi_{\lambda_0}), \psi\rangle > 0$. The existence of such a function ψ can be proved using arguments similar to those in Lemma 3.4. One can easily prove that $I(\varphi_{\lambda_0} + \frac{\psi}{n}) > 0$ for n large enough. Moreover, for such n we can find $0 < t_n, s_n < \frac{1}{n}$ such that

$$E_V\left(\varphi_{\lambda_0} + \frac{\psi}{n}\right) = \frac{1}{n}\left\langle E_V'(\varphi_{\lambda_0} + t_n\psi), \psi\right\rangle$$

and

$$I\left(\varphi_{\lambda_0} + \frac{\psi}{n}\right) = \left\langle I'(\varphi_{\lambda_0} + s_n\psi), \psi\right\rangle$$

Hence $E_V(u_n) \to \lambda_0$. Since φ_{λ_0} satisfies $I(\varphi_{\lambda_0}) = 0$ it is clear that the $\inf_{\mathcal{M}^+} E_V$ is not achieved. Finally for any L^p-normalized function u satisfying $E_V(u) = I(u) = 0$ one has

$$\sup_{\lambda\in\mathbb{R}} \mu(\lambda) = 0 = E_V(u) = E_{V-\lambda_0 m}(u) - \lambda_0 \oint_{\Gamma_1} \sigma|u|^p\, d\rho \geq \mu(\lambda_0) = 0,$$

so u achieves $\mu(\lambda_0)$. Thus $u = c\varphi_{\lambda_0}$ for some constant c and the result follows with $\lambda_* := \lambda_0$. □

5. A sufficient condition for the existence of principal eigenvalues

Let us consider the following eigenvalue problem with mixed Dirichlet–Newmann boundary conditions:

$$\begin{cases} -\Delta_p u + V(x)|u|^{p-2}u = \lambda|u|^{p-2}u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_1, \\ u = 0 & \text{on } \Gamma_2, \end{cases} \tag{N}$$

which correspond to the case $\sigma = 0$ and $m \equiv 1$ in problem (P). Denote

$$\lambda_1^{\mathcal{N},\mathcal{D}}(V) = \inf\left\{E_V(u) \;\middle|\; u \in W, \int_\Omega |u|^p\, dx = 1\right\}.$$

Noticing that $\mu(0) = \lambda_1^{\mathcal{N},D}(V)$ we have the following trivial consequence of Theorem 4.1:

Proposition 5.1. *Assume that $\alpha \geq 0$.*

(1) If $\lambda_1^{\mathcal{N},D}(V) > 0$ then problem (P) possesses exactly two principal eigenvalues of different sign except in the cases **(i)** *$m \geq 0$ in Ω and $\sigma \geq 0$ on Γ_1 where there is exactly one principal eigenvalue, which is positive, or in the case* **(ii)** *$m \leq 0$ in Ω and $\sigma \leq 0$ on Γ_1 where there is exactly one principal eigenvalue, which is negative.*

(2) If $\lambda_1^{\mathcal{N},D}(V) = 0$ then problem (P) possesses a unique nontrivial eigenvalue (which is positive) if and only if $m^+ \not\equiv 0$ in Ω or $\sigma^+ \not\equiv 0$ on Γ_1 and

$$d := \int_\Omega m|\varphi_0|^p\, dx + \oint_{\Gamma_1} \sigma|\varphi_0|^p\, d\rho < 0,$$

where φ_0 is the positive eigenfunction associated to $\lambda_1^{\mathcal{N},D}(V)$ of L^p-norm equal to 1.

(3) If $\lambda_1^{\mathcal{N},D}(V) = 0$ then problem (P) possesses a unique nontrivial eigenvalue (which is negative) if and only if $m^- \not\equiv 0$ in Ω or $\sigma^- \not\equiv 0$ on Γ_1 and $d > 0$.

(4) If $\lambda_1^{\mathcal{N},D}(V) = 0$ and $d = 0$ then $\lambda = 0$ is the unique principal eigenvalue of problem (P).

Proof. One uses that

$$\mu'(0) = -\int_\Omega m|\varphi_0|^p\,dx - \oint_{\Gamma_1} \sigma|\varphi_0|^p\,d\rho = -d.$$

(1) In this case $\alpha = \sup_{\lambda\in\mathbb{R}} \mu(\lambda) \geq \mu(0) = \lambda_1^{\mathcal{N},D}(V) > 0$ and the situations **(i)** and **(ii)** correspond to those of case (1) of Theorem 4.1.

(2) In this case we have $\lambda_{-1} = \mu(0) = \lambda_1^{\mathcal{N},D}(V) = 0$. Since $\mu'(0) = -d > 0$, by the concavity of the curve $\lambda \to \mu(\lambda)$ it must be $\alpha > 0$ and the result follows from the case (2)(a) of Theorem 4.1.

(3) Similar to case (2).

(4) If $\mu(0) = \lambda_1^{\mathcal{N},D}(V) = 0$ and $d = 0$ then $\alpha = 0$ and therefore $\lambda = 0$ is the unique principal eigenvalue of problem (P). □

Remark 5.2. When $V \equiv 0$ and $\Gamma_2 = \emptyset$ then $\lambda_1^{\mathcal{N},D}(V) = 0 = \mu(0)$ and we can choose $\varphi_0 = cst$. Our results (2) and (3) in Proposition 5.1 generalize for the p-laplacian operator the result of [23].

6. On the coerciveness of the restricted functional

Let us show in this section that $\alpha > 0$ is a sufficient condition for the coerciveness of E_V under the constrain $\mathcal{M}^\pm$.

Proposition 6.1. *If $\alpha > 0$ then, for any $M \in \mathbb{R}$, the set*

$$\left\{u \in \mathcal{M}^+ \;\middle|\; E_V(u) \leq M\right\}$$

is bounded. A similar result can be stated for $-E_V$ on $\mathcal{M}^-$.

Proof. Assume by contradiction that for some $M \in \mathbb{R}$ there is sequence (u_n) unbounded in $\mathcal{M}^+$ and satisfying $E_V(u_n) \leq M$. Then, there exits a subsequence of $v_n = \frac{u_n}{\|u_n\|}$, still denoted as v_n, that converges to some v_0 weakly in W and strongly in $L^p(\Omega) \cap L^p(\partial\Omega, \rho)$. Since $I(v_n) = \frac{1}{\|u_n\|^p} \to 0$, we deduce that $I(v_0) = 0$ and

$$\int_\Omega |\nabla v_0|^p dx + \int_\Omega V|v_0|^p dx \leq \liminf_{n\to+\infty} E_V(v_n) = \liminf_{n\to+\infty} \frac{E_V(u_n)}{\|u_n\|^p} = 0. \tag{6.1}$$

We claim that $v_0 \neq 0$, otherwise

$$0 \leq \liminf_{n\to+\infty} \int_\Omega |\nabla v_n|^p\,dx = \liminf_{n\to+\infty} \left(E_v(v_n) - \int_\Omega V|v_n|^p\,dx\right) \leq 0$$

and consequently $v_n \to 0$ strongly in W, which is impossible since $\|v_n\| = 1$. Hence $\frac{v_0}{\|v_0\|_p}$ is an admissible function in the definition of α and consequently

$$\alpha \leq E_V\left(\frac{v_0}{\|v_0\|_p}\right) \leq 0,$$

gives a contradiction with the assumption $\alpha > 0$. □

Next let us justify that $\alpha > 0$ is "almost" a necessary for E_V to be coercive on $\mathcal{M}^\pm$. The case $\Gamma_1 = \emptyset$ was already considered in [13].

Proposition 6.2. *Assume that $\sigma^+ \not\equiv 0$ (resp. $\sigma^- \not\equiv 0$). If $\alpha < 0$ then E_V (resp. $-E_V$) is unbounded from below on M^+ (resp. on M^-).*

Proof. Let $0 \leq u_0 \in \mathcal{G}$ realize α. We distinguish four cases:

(a) If $m^+ \not\equiv 0$, we pick $0 \leq w \in W_0^{1,p}(\Omega)$ such that $0 \not\equiv w$ and $supp\, w \subset \Omega^+$. Thus, for each n, there is $0 < s_n < \frac{1}{n}$ such that

$$I\left(u_0 + \frac{w}{n}\right) = \frac{1}{n} \int_{\Omega^+} m(u_0 + s_n w)^{p-2}(u_0 + s_n w) w\, dx + \oint_{\Gamma_1} \sigma u_0^p\, d\rho > 0.$$

Hence

$$\frac{E_V(u_0 + \frac{w}{n})}{I(u_0 + \frac{w}{n})} \to -\infty.$$

(b) If $m \leq 0$ and $mu_0 \not\equiv 0$, we pick a function $0 \leq w \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that $0 \not\equiv w \leq u_0$.

$$I\left(u_0 - \frac{w}{n}\right) = \int_{\Omega} m\left|u_0 - \frac{w}{n}\right|^p dx + \oint_{\Gamma_1} \sigma u_0^p\, d\rho > I(u_0) = 0$$

Then

$$\frac{E_V(u_0 - \frac{w}{n})}{I(u_0 - \frac{w}{n})} \to -\infty.$$

(c) If $m \leq 0$, $mu_0 \equiv 0$ and the adherence of Ω^0 intersects Γ_1^+ we take a function $0 \leq w \in W$ with support in Ω^0 and such that $supp\, \gamma(w) \subset \Gamma_1^+$ (we construct w as in Lemma 3.4). Then

$$I\left(u_0 + \frac{w}{n}\right) = \frac{1}{n^p} I(w) > 0, \qquad \frac{E_V(u_0 + \frac{w}{n})}{I(u_0 + \frac{w}{n})} \to -\infty. \tag{6.2}$$

(d) If $m \leq 0$, $mu_0 \equiv 0$ and we are not in the previous cases, then we must have $\overline{\Omega^0} \subset \Gamma_1^0$ because $I(u_0) = \oint_{\Gamma_1} \sigma u_0^p\, d\rho = 0$. In this case we take a function $0 \leq w \in W$ from Lemma 3.4 such that $\int_{\Gamma_1} \sigma w^p d\rho > 0$ and having support in Ω^- and also $supp\, \gamma(w) \subset \Gamma_1^+$. Then we also have (6.2). □

If $\alpha = 0$ we have a less general result:

Proposition 6.3. *Assume that $\sigma^+ \not\equiv 0$. If $\alpha = 0$ and $u_n \geq 0$ is a sequence in $\mathcal{M}^+$ such that $\lambda_N(\{x \in \Omega \mid u_n(x) > 0\}) \to 0$ and $(E_V(u_n))$ is bounded then (u_n) is bounded in W. A similar result can be stated in the case $\sigma^- \not\equiv 0$ for $-E_V$ over $\mathcal{M}^-$.*

Proof. Assume by contradiction that the sequence $(u_n)_n$ is unbounded. Consider $v_n = \frac{u_n}{\|u_n\|}$, which converges weakly to some $v_0 \in W$ and strongly in $L^p(\Omega) \cap L^p(\partial\Omega, \rho)$. Thus, passing to the limit, we get $I(v_0) = 0$ and

$$\lim_{n\to+\infty} E_V(v_n) = \lim_{n\to+\infty} \frac{E_V(u_n)}{||u_n||^p} = 0.$$

Moreover

$$1 + \int_{\Omega} V|v_0|^p\, dx = 1 + \lim_{n\to+\infty} \int_{\Omega} V|v_n|^p\, dx = \lim_{n\to+\infty} E_V(v_n) = 0.$$

We deduce in particular that $v_0 \not\equiv 0$. Hence $\frac{v_0}{\|v_0\|_p}$ is admissible in the definition of α. By Proposition 4.1(2)(b) $v_0 > 0$ on Ω, in contradiction with

$$\lambda_N\big(\{x \in \Omega \mid v_n(x) > 0\}\big) = \lambda_N\big(\{x \in \Omega \mid u_n(x) > 0\}\big) \to 0. \qquad \square$$

It is not difficult to construct examples of V and m such that $\alpha = \alpha(V, m, \sigma) = 0$ by adapting some ideas of [13].

Example 6.3.1. Let Ω be a bounded domain in $\mathbb{R}^N$ and $B_0 \subset \Omega$ be an open subset such that $\|\psi_1\|^2_{2,B_0} = \frac{1}{2}$, where ψ_1 is an eigenfunction associated to the first eigenvalue of

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda\sigma |u|^{p-2}u & \text{on } \Gamma_1, \\ u = 0 & \text{on } \Gamma_2, \end{cases}$$

satisfying $\int_\Omega \psi_1^p\, dx + \oint_{\Gamma_1} \sigma\psi_1^p\, d\rho = 1$. Thus, $\lambda_1 = \lambda_1$ $(V \equiv 0,\ m \equiv 1, \sigma)$ as defined in Theorem 4.1. Set

$$m = \begin{cases} a & \text{in } \Omega \backslash B_0, \\ b & \text{in } B_0 \end{cases} \tag{6.3}$$

for some a, b to be determined later. We have $\int_\Omega m\psi_1^p\, dx + \oint_{\Gamma_1} \sigma\psi_1^p\, d\rho = 0$ by choosing $\frac{a+b}{2} - 1 = -(\int_\Omega \psi_1^p)^{-1}$. Let

$$V = \begin{cases} \lambda_1(a-1), & \text{in } \Omega \backslash B_0 \\ \lambda_1(b-1), & \text{in } B_0. \end{cases} \tag{6.4}$$

If u is such that $\int_\Omega m|u|^p\, dx + \oint_{\Gamma_1} \sigma|u|^p\, d\rho = 0$ then

$$\begin{aligned} E_V(u) &= \int_\Omega |\nabla u|^p\, dx + \lambda_1(b-1)\|u\|^p_{p,B_0} + \lambda_1(a-1)\|u\|^p_{p,\Omega\backslash B_0} \\ &= \int_\Omega |\nabla u|^p\, dx - \lambda_1\Big(\int_\Omega |u|^p\, dx + \oint_{\Gamma_1} \sigma|u|^p\, d\rho\Big) \geq 0 \end{aligned}$$

Moreover, $E_V(u) = 0$ holds precisely for multiples of ψ_1. Therefore $\alpha(V, m, \sigma) = 0$.

Example 6.3.2. We give another example where $\alpha(V, m, \sigma) = 0$ in the case $m \equiv 0$. Let us assume here that Γ_1 is not a single point if $N = 1$. For a given $V \in L^\infty(\Omega)$, let us take $\Gamma_0 \subset \Gamma_1$ a subset such that $\|\psi_0\|^p_{p,\Gamma_0} = \frac{1}{2}$, where ψ_0 is an eigenfunction associated to the first eigenvalue of problem (N)

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_1, \\ u = 0 & \text{on } \Gamma_2, \end{cases}$$

satisfying $\oint_{\Gamma_1} \psi_0^p\, d\rho = 1$. Thus, $\lambda_1 = \lambda_1^{\mathcal{N},\mathcal{D}}(0)$. Set

$$\sigma = \begin{cases} 1 & \text{in } \Gamma_1 \backslash \Gamma_0, \\ -1 & \text{in } \Gamma_0, \end{cases} \tag{6.5}$$

then we have $\oint_{\Gamma_1} \sigma\psi_0^p\, d\rho = 0$. Set $V := -\lambda_1^{\mathcal{N},\mathcal{D}}(0)$. If u is such that $\oint_{\Gamma_1} \sigma|u|^p\, d\rho = 0$ then

$$E_V(u) = \int_\Omega |\nabla u|^p\, dx + \int_\Omega V|u|^P\, dx \geq 0$$

and $E_V(u) = 0$ holds precisely for multiples of ψ_0. Therefore $\alpha(V, 0, \sigma) = 0$.

7. Simplicity and isolation of the principal eigenvalues

Here below we prove two properties of the principal eigenvalues of problem (P).

Proposition 7.1 *(Simplicity). Assume that $u, v \in W$ are two eigenfunctions of problem (P) associated respectively to λ and β. Assume also that $u > 0$ and $v > 0$ in Ω. If $\beta \geq \lambda$ (resp. $\beta \leq \lambda$) when $I(u) \geq 0$ (resp. when $I(u) \leq 0$) then $u = cv$ for some $c > 0$ and $\lambda = \beta$.*

In particular, if $\alpha \geq 0$, the principal eigenvalues λ_1 and λ_{-1} are simple. There are no other principal eigenvalues.

Proof. We apply Picone's Identity of Lemma 2.1 to u and $v + \eta$. After integration and letting $\eta \to 0$ we find

$$\begin{aligned}
0 \leq \int_\Omega L(u,v)\, dx &= \int_\Omega R(u,v)\, dx \\
&= \int_\Omega \left[|\nabla u|^p - |\nabla v|^{p-2} \nabla\left(\frac{u^p}{v^{p-1}}\right) \nabla v \right] dx \\
&= \lambda\left(\int_\Omega m|u|^p\, dx + \oint_{\Gamma_1} \sigma u^p\, d\rho \right) - \int_\Omega V u^p\, dx \\
&\quad + \int_\Omega (V - \beta m) u^p\, dx - \beta \oint_{\Gamma_1} \sigma u^p\, d\rho \\
&= (\lambda - \beta)\left(\int_\Omega m u^p\, dx + \oint_{\Gamma_1} \sigma u^p\, d\rho \right).
\end{aligned}$$

If $\int_\Omega m u^p\, dx + \oint_{\Gamma_1} \sigma u^p\, d\rho > 0$ (resp. $\int_\Omega m u^p\, dx + \oint_{\Gamma_1} \sigma u^p\, d\rho < 0$) and $\beta \geq \lambda$ (resp. $\beta \leq \lambda$) then $L(u,v) = 0$. Hence $u = cv$ for some $c > 0$ and therefore $\alpha = \beta$. If $\int_\Omega m u^p\, dx + \oint_{\Gamma_1} \sigma u^p\, d\rho = 0$ then $L(u,v) = 0$ and we conclude again that $u = cv$ and $\alpha = \beta$. □

Proposition 7.2 *(Isolation). Assume that $\alpha \geq 0$. Then the principal eigenvalues are isolated in the spectrum of (P).*

Proof. We only prove that λ_1 is isolated, an analogous proof can be given for λ_{-1}. We can use similar arguments to those of [12]. First one gets an a-priori estimate of the measure of any nodal set $\mathcal{N}$ of a non-principal eigenfunction u associated to λ by using Sobolev and trace embeddings. We recall that a *nodal domain of* u is a connected component of $\Omega \setminus \{x \in \Omega \mid u(x) = 0\}$. This estimate will read as follows

$$\lambda_N(\mathcal{N})^{\frac{p^*-p}{p^*}} + \rho(\overline{\mathcal{N}} \cap \Gamma_1)^{\frac{p_*-p}{p_*}} \geq (|\lambda| c_1 + c_2)^{-1} \tag{7.1}$$

for some positive constants c_1, c_2 depending only upon m, σ, V and Ω. We explain briefly how to prove (7.1). First observe that, since $u \in W \cap C(\overline{\Omega})$, then $u_{|\mathcal{N}} \in W_{\mathcal{N}}$. Hence the function w defined as $w(x) = u(x)$ if

$x \in \mathcal{N}$ and $w(x) = 0$ if $x \in \Omega \setminus \mathcal{N}$ belongs to W. Assume that $1 < p < N$. Using w as a test function in the weak equation satisfied by u we find

$$\int_{\mathcal{N}} |\nabla v|^p \, dx + \oint_{\overline{\mathcal{N}} \cap \Gamma_1} |u|^p \, d\rho \le C\big(\|u\|^p_{p^*,\mathcal{N}} \lambda_N(\mathcal{N})^{\frac{p^*-p}{p^*}} + \|u\|^p_{p_*,\overline{\mathcal{N}}\cap\Gamma_1} \rho(\overline{\mathcal{N}} \cap \Gamma_1)^{\frac{p_*-p}{p_*}}\big)$$

by Hölder inequality, with $C = (|\lambda|(\|m\|_\infty + \|\sigma\|_\infty) + \|V\|_\infty + 1)$. On the other hand, using Sobolev's and trace embeddings we have that

$$\int_{\mathcal{N}} |\nabla v|^p \, dx + \oint_{\overline{\mathcal{N}} \cap \Gamma_1} |u|^p \, d\rho \ge D\big(\|v\|^p_{p^*,\mathcal{N}} + \|u\|^p_{p_*,\overline{\mathcal{N}}\cap\Gamma_1}\big)$$

for some new constant $D = D(N, p, \Omega)$ and the result follows. In the case $p \ge N$ we will proceed similarly.

Assume now by contradiction that there exists (λ_n, u_n) a sequence of eigenvalues and eigenfunctions such that $\lambda_n \searrow \lambda_1$.

(a) Assume $\alpha > 0$. If $I(u_n) = 0$ then $0 < \alpha \le E_V(u_n) = \lambda_n I(u_n) = 0$, a contradiction. Since $\lambda_n > \lambda_1$ it follows that $I(u_n) > 0$, otherwise

$$-\lambda_{-1} \le -\frac{E_V(u_n)}{I(u_n)} = -\lambda_n < -\lambda_1,$$

a contradiction. Thus $v_n := \frac{u_n}{I(u_n)^{1/p}} \in \mathcal{M}^+$ is such that $E_V(v_n) = \lambda_n \searrow \lambda_1$. By Proposition 6.1 the sequence v_n is bounded in W so there exists $v_0 \in W$ such that $v_n \rightharpoonup u_0$ weakly in W and strongly in $L^p(\Omega) \cap L^p(\partial\Omega, \rho)$. Using the fact that $\lambda_1 \le E_V(v_0) \le \lim_{n\to+\infty} E_v(v_n) = \lambda_1$, we conclude that $u_0 = \pm\varphi_1$. Remember that, by Proposition 3.2, $\varphi_1 > 0$ in Ω and $\varphi_1 > 0$ on Γ_1. In the case $v_0 = \varphi_1$, let Ω_n^- be a negative nodal domain of v_n. From the convergence in measure of v_n towards v_0 we conclude that $\lambda_N(\Omega_n^-) + \rho(\overline{\Omega_n^-} \cap \Gamma_1) \to 0$ as $n \to +\infty$, in contradiction with (7.1). In the case $v_0 = -\varphi_1$ we will argue similarly.

(b) Assume $\alpha = 0$. If $I(u_n) = 0$ then, by the result of Theorem 4.1(2)(b), u_n will be a multiple of φ_{λ_*} which is impossible as $\lambda_n \ne \lambda_*$. Arguing as in the previous case we have that $I(u_n) > 0$. Let us prove that the sequence $v_n = \frac{u_n}{I(u_n)^{1/p}} \in \mathcal{M}^+$ is bounded in W. If not, take $w_n = \frac{v_n}{\|v_n\|}$ and w_0 a weak limit of a subsequence converging strongly in $L^p(\Omega) \cap L^p(\partial\Omega, \rho)$. Then $E_V(w_0) = I(w_0) = 0$. Moreover, since

$$\int_{\Omega} |\nabla w_0|^p \, dx \le \liminf_{n\to+\infty} \Big(E_V(w_n) - \int_{\Omega} V|w_n|^p \, dx\Big) = 0,$$

then $w_0 \ne 0$ otherwise $w_n \to w_0 = 0$ strongly in W, which will contradict that $\|w_n\| = 1$. So $\frac{w_0}{\|w_0\|_p} \in \mathcal{G}$ is a function where the value $\alpha = 0$ is achieved. By Theorem 4.1(2)(b), w_0 is an eigenfunction for the principal eigenvalue λ_*. Hence we will reach a contradiction using, as before, the estimate of the measure of the nodal domains of w_n. □

8. Existence of nonprincipal eigenvalues

Our aim is to prove the existence of a sequence of eigenvalues for problem (P). In some cases we can even establish the existence of *two sequences* of eigenvalues, one converging to $+\infty$ and the other to $-\infty$.

For simplicity, we will assume that either $\sigma^+ \not\equiv 0$ or $m^+ \not\equiv 0$ and we will prove the existence of a sequence of eigenvalues going to $+\infty$ by constructing a sequence of critical values of E_V restricted to $\mathcal{M}^+$ via the

Ljusternik–Schnirelmann critical theory on C^1 manifolds (see [21] or [22]). In order to do so, we define for any $k \in \mathbb{N}^*$,

$$\mathcal{A}_k \stackrel{\text{def}}{=} \{K \subset \mathcal{M}^+ \mid K \text{ symmetric, compact and } i(K) = k\},$$

where $i(K)$ denotes Krasnoselski's genus of K on $W^{1,p}(\Omega)$. Let us first investigate when $\mathcal{A}_k$ is a nonempty set.

Lemma 8.1. *Assume that either $\sigma^+ \not\equiv 0$ or $m^+ \not\equiv 0$. In the case $m^+ \equiv 0$ assume furthermore that $N > 1$. Then $\mathcal{A}_k \neq \emptyset$ for all $k \in \mathbb{N}^*$.*

Proof. Let $k \in \mathbb{N}^*$ be fixed. If $m^+ \not\equiv 0$, we can construct functions $e_i \in W_0^{1,p}(\Omega)$, $i = 1, \cdots, k$, such that for $i \neq j$, $\mathit{supp}\, e_i \cap \mathit{supp}\, e_j = \emptyset$ and $\int_\Omega m|e_i|^p\, dx > 0$ by regularizing, for instance, characteristic functions on small disjoint balls of Ω that have nonempty intersection with Ω^+. Then

$$F := \left\{\sum_{i=1}^{k} \alpha_i e_i \mid \alpha_i \in \mathbb{R}\right\} \cap \mathcal{M}^+ \in \mathcal{A}_k.$$

If $m^+ \equiv 0$ we show how to construct functions $e_i \in W$ with $\mathit{supp}\, e_i \cap \mathit{supp}\, e_j = \emptyset$ and $\mathit{supp}\, \gamma(e_i) \cap \mathit{supp}\, \gamma(e_j) = \emptyset$, if $i \neq j$. To be more precise, pick k disjoint balls B_i of Ω such that $S_i := \overline{B_i} \cap \Gamma_1 \neq \emptyset$ and such that $\rho(S_i \cap \{x \in \Gamma_1 \mid \sigma(x) > 0\}) > 0$ for all $i = 1, \cdots k$. Notice that, in order to assure the existence of such disjoint supports on Γ_1, we need to assure that the boundary set Γ_1 is not a single point, and therefore we exclude the case of dimension $N = 1$. Then take $\psi_i \in C^\infty(\Gamma_1)$ a nonnegative function such that $a_i := \oint_{\Gamma_1} \sigma|\psi_i|^p\, d\rho > 0$, by regularizing the characteristic function of $S_i \cap \{x \in \Gamma_1 \mid \sigma(x) > 0\}$. Let $v_i \in C^\infty(\Omega)$ such that $\gamma(v_i) = \psi_i$. Then multiply v_i by a regularization of the characteristic function of a small $B_i^0 \subset B_i$ chosen in such a way that

$$\lambda_N\left(B_i^0\right) < \frac{a_i}{2\|m\|_\infty \|v_i\|_\infty^p},$$

to get a function $e_i \in W$ with trace $\gamma(e_i) = \psi_i$, $\mathit{supp}\, e_i \cap \mathit{supp}\, e_j = \emptyset$, if $i \neq j$, and such that $\int_\Omega m|e_i|^p\, dx + \oint_{\Gamma_1} \sigma|\psi_i|^p\, d\rho > 0$. □

Theorem 8.2. *Let us assume $\alpha > 0$ and that either $\sigma^+ \not\equiv 0$ or $m^+ \not\equiv 0$. In the case $m^+ \equiv 0$ assume that $N > 1$. Define, for any $k \in \mathbb{N}^*$,*

$$\lambda_k \stackrel{\text{def}}{=} \inf_{K \in \mathcal{A}_k} \max_{u \in K} E_V(u). \tag{8.1}$$

Then $(\lambda_k)_k$ is a nondecreasing sequence of eigenvalues of the problem (P) such that $\lim_{k\to+\infty} \lambda_k = +\infty$. Moreover

$$\lambda_2 = \inf\{\lambda \mid \lambda > \lambda_1 \textit{ and } \lambda \textit{ is an eigenvalue of } (P)\}. \tag{8.2}$$

Proof. In order to apply the Ljusternik–Schnirelmann critical theory on C^1 manifolds it suffices to prove that the restriction of E_V to $\mathcal{M}^+$ satisfies the Palais–Smale condition at the level λ_k. This will imply (cf. [21, Theorem 3.54] or [22]) both that λ_k is an eigenvalue associated to problem (P), and also $\lim_{k\to+\infty} \lambda_k = +\infty$.

Let (u_n) be a Palais–Smale sequence in $\mathcal{M}^+$ for E_V, i.e.

$$\begin{aligned} &(\text{PS1}) \quad E_V(u_n) \longrightarrow c \\ &(\text{PS2}) \quad \left|\left\langle E_V'(u_n), \xi\right\rangle\right| \le \varepsilon_n \|\xi\| \quad \text{for all } \xi \in T_{u_n}\mathcal{M}^+. \end{aligned}$$

For any $w \in W$, it is clear that

$$a_n(w) = w - \Big[\int\limits_{\Omega} m|u_n|^{p-2}u_n w + \oint\limits_{\Gamma_1} \sigma|u_n|^{p-2}u_n w\Big]u_n \in T_{u_n}\mathcal{M}^+.$$

Hence, taking $\xi = a_n(w)$ in (PS2), we get

$$\big|\big\langle E'_V(u_n), w\big\rangle - A_n E_V(u_n)\big| \le \varepsilon_n \|w - A_n u_n\|, \tag{8.3}$$

where

$$A_n := \int\limits_{\Omega} m|u_n|^{p-2}u_n w + \oint\limits_{\Gamma_1} \sigma|u_n|^{p-2}u_n w.$$

By Proposition 6.1, the sequence u_n is bounded and therefore, there exits a subsequence still denoted u_n, such that $u_n \to u_0$ weakly in W and strongly in $L^p(\Omega) \cap L^p(\partial\Omega, \rho)$ for some u_0. Choosing $v = u_n - u_0$ in (8.3) and passing to the limit we obtain

$$\lim_{n\to+\infty} \int\limits_{\Omega} |\nabla u_n|^{p-2}\nabla u_n \nabla(u_n - u_0)dx = 0.$$

Applying Lemma 2.2 and Hölder inequality, one easily derives that $\nabla u_n \to \nabla u_0$ in $L^p(\Omega)$ and consequently $u_n \to u_0$ in $W^{1,p}(\Omega)$.

Let us prove the characterization of λ_2. Of course $\lambda_2 > \lambda_1$ otherwise one will have $i(K_{\lambda_1}) \ge 2$, where K_{λ_1} is the set of eigenfunctions in $\mathcal{M}^+$ associated to λ_1. This is a contradiction because λ_1 is simple. Assume now by contradiction that there exists an eigenvalue λ of problem (P) between λ_1 and λ_2 and let $v \in \mathcal{M}^+$ a corresponding eigenfunction. Since v changes sign by Proposition 7.1, by multiplying Eq. (P) by $v^\pm$ if follows that $E_v(v^\pm) = \lambda I(v^\pm)$. It comes from this identity that $I(v^\pm) \ne 0$ otherwise $v^\pm \in \mathcal{G}$, and $E_V(v^\pm) = 0$ and therefore $\alpha \le 0$, a contradiction. Also $I(v^\pm) > 0$ otherwise

$$-\lambda_{-1} \le \frac{E_V(v^\pm)}{-I(v^\pm)} = -\lambda < -\lambda_1,$$

which is absurd. Consider then the set

$$A = \big\{av^+ + bv^- \mid a, b \in \mathbb{R}\big\} \cap \mathcal{M}^+.$$

Using the fact that the supports of v^+ and v^- are disjoint, one can easily prove that A is homomorphic to a sphere of $\mathbb{R}^2$ and therefore $i(A) = 2$. Hence

$$\lambda_2 \le \max_{v\in A} E_V(v) = \lambda,$$

a contradiction. □

A trivial corollary of Theorem 8.2 is

Corollary 8.3. *Assume that $\sigma^- \not\equiv 0$ or $m^- \not\equiv 0$. In the case $m^- \equiv 0$ assume that $N > 1$. Assume also $\alpha > 0$. Then there exists a nonincreasing sequence λ_{-k} of eigenvalues of the problem (P) defined as*

$$\lambda_{-k} \stackrel{\text{def}}{=} \inf_{K\in\mathcal{B}_k} \max_{u\in K} -E_V(u),$$

where $\mathcal{B}_k \stackrel{\text{def}}{=} \{K \subset \mathcal{M}^- \mid K$ symmetric, compact and $i(K) = k\}$, satisfying $\lim_{k\to+\infty} \lambda_{-k} = -\infty$.

Remark 8.4. The characterization of λ_2 as the second eigenvalue on the right of (P) was first proved, for the Dirichlet problem and $V \equiv 0$, in [5]. In [13] a second characterization of the second eigenvalue on the right was found for the Dirichlet problem with V, m indefinite.

In the case $\alpha = 0$, it is more delicate to construct nonprincipal eigenvalues by variational methods since *the* (PS) *fails at the level* $\lambda_1 = \lambda_{-1} = \lambda_*$. Indeed, notice that the eigenvalue λ_* of Theorem 4.1 *is not achieved* neither at $\mathcal{M}^+$ nor $\mathcal{M}^-$, but at any function of $\mathcal{G}$. Although one can define the Ljusternik–Schnirelmann sequence $\{\lambda_k\}_k$ as above, one needs a compactness condition to prove that those values are critical values of E_V restricted to the manifold $\mathcal{M}^+$. We will prove in the next lemma a weaker compactness condition for all levels greater than λ_*.

Lemma 8.5. *Assume that $\alpha = 0$. Then E_V satisfies the Palais–Smale Condition of Cerami at level c ($(PSC)_c$ for short) on $\mathcal{M}^+$ for any $c > \lambda_*$.*

Proof. Let us prove the $(PSC)_c$ condition for any $c > \lambda_*$. Let u_n be a $(PSC)_c$ sequence in $\mathcal{M}^+$ for E_V, i.e., there exists $\varepsilon_n \to 0$ such that

$$\begin{aligned} &(\text{PSC1}) \quad E_V(u_n) \longrightarrow c \\ &(\text{PSC2}) \quad \left|\left\langle E_V'(u_n), \xi\right\rangle\right| \le \frac{\varepsilon_n}{1+\|u_n\|}\|\xi\| \quad \text{for all } \xi \in T_{u_n}\mathcal{M}^+. \end{aligned}$$

Let us assume by contradiction that (u_n) is unbounded and set $v_n = \frac{u_n}{\|u_n\|}$. Up to a subsequence, there is some v_0 such that $v_n \rightharpoonup v_0$ in W and $v_n \to v_0$ in $L^p(\Omega) \cap L^p(\partial\Omega, \rho)$. We choose $\xi = a_n(v_n - v_0)$ in $(PSC2)$ and divide it by $\|u_n\|^{p-1}$ to obtain

$$\left|\left\langle E_V'(v_n), v_n - v_0\right\rangle - B_n E_V(u_n)\right| \le \frac{\varepsilon_n \|u_n\|}{1+\|u_n\|}\left\|\frac{v_n - v_0}{\|u_n\|^p} - B_n v_n\right\|, \tag{8.4}$$

where

$$B_n = \int_\Omega m|v_n|^{p-2} v_n (v_n - v_0) + \oint_{\Gamma_1} \sigma |v_n|^{p-2} v_n (v_n - v_0).$$

By letting $n \to \infty$ and using the (S^+) property of the p-laplacian (cf. Lemma 2.2) we get that $v_n \to v_0$ in W and in particular $v_0 \ne 0$. Moreover $E_V(v_0) = 0$ and $\int_\Omega m|v_0|^p\,dx + \oint_{\Gamma_1} \sigma|v_0|^p\,d\rho = 0$, then v_0 achieves α. By Theorem 4.1(2)(b), v_0 has definite sign and is an eigenfunction of (P). Furthermore, when choosing $\xi = a_n(w)$ by any $w \in W$ and letting $n \to \infty$ in (8.4), we find that v_0 is an eigenfunction associated to c, in contradiction with $c > \lambda_*$. Therefore the sequence u_n is bounded and, again by the (S^+) property $(u_n)_n$ is convergent in W up to a subsequence. □

Remark 8.6. It is not clear in our context if the 2nd $L - S$ infmax-value is *strictly greater* than λ_*. To our knowledge, the compactness condition (either (PS) or (PSC)) is needed to prove both that the second $L-S$ infmax-value is greater than λ_1 and that it is a critical value of E_V restricted to $\mathcal{M}^+$. We fail to give an answer to any of these questions.

Next we construct a nonprincipal eigenvalue using the ideas of [13]. We refer the reader to this paper for complete details relative of the following theorem.

Theorem 8.7. *Assume that either $\sigma^+ \not\equiv 0$ or $m^+ \not\equiv 0$. In the case $m^+ \equiv 0$ assume that $N > 1$. Let us also assume that $\alpha = 0$.*

(i) Let $\mathcal{L} := \{h \in C([0,1], \mathcal{M}^+) \mid h(0) \geq 0,\ h(1) \leq 0\}$*. Then the value*

$$\mu \stackrel{\text{def}}{=} \inf_{h \in \mathcal{L}} \max_{u \in h([0,1])} E_V(u),$$

satisfies $\mu > \lambda_*$*.*

(ii) For any $u_1 \in \mathcal{M}^+$*,* $u_1 \geq 0$*, such that* $E_V(u_1) < \mu$*,*

$$\lambda_2 \stackrel{\text{def}}{=} \inf_{h \in \mathcal{H}} \max_{u \in h([0,1])} E_V(u),$$

where $\mathcal{H} := \{h \in C([0,1], \mathcal{M}^+) \mid h(0) = u_1,\ h(1) = -u_1\}$*, is an eigenvalue for problem (P).*

(iii) $\mu = \lambda_2$*.*

(iv) λ_2 *is the first nonprincipal eigenvalue of problem (P) satisfying (8.2).*

Proof. (i) First notice that $\mathcal{L} \neq \emptyset$ as one can always pick two functions in $\mathcal{M}^+$ of definite sign, $v_1 \geq 0$, $v_2 \leq 0$, with disjoint supports and construct the path $\gamma_1(t) := \frac{t^{\frac{1}{p}} v_1 - (1-t)^{\frac{1}{p}} v_2}{I(t^{\frac{1}{p}} v_1 - (1-t)^{\frac{1}{p}} v_2)^{1/p}}$ for $t \in [0,1]$. Then $\gamma_1 \in \mathcal{L}$. Let us assume by contradiction that equality $\lambda_* = \mu$ holds. Then there exists $h_k \in \mathcal{L}$ such that $\max_{t \in [0,1]} E_V(h_k(t)) \to \lambda_*$ when $k \to \infty$. For every k one can find $t_k \in [0,1]$ satisfying

$$I\big(h_k(t_k)^+\big) = I\big(h_k(t_k)^-\big) = \frac{1}{2}. \tag{8.5}$$

We set $u_k = h_k(t_k)$. Notice, from (8.5), that $2^{\frac{1}{p}} u_k^{\pm} \in \mathcal{M}^+$ so that $E_V(u_k^{\pm}) \geq \frac{1}{2}\lambda_*$. Hence

$$\frac{1}{2}\lambda_* \leq E_V\big(u_k^{\pm}\big) = E_V(u_k) - E_V\big(u_k^{\mp}\big) \leq \max_{t \in [0,1]} E_V\big(\gamma_k(t)\big) - \frac{\lambda_*}{2} \to \frac{\lambda_*}{2}$$

so that

$$\lim_{k \to +\infty} E_V\big(u_k^{\pm}\big) = \frac{\lambda_*}{2}. \tag{8.6}$$

Let us show that the sequence (u_k) is bounded in W. Assume by contradiction that the sequence (u_k) is unbounded and set $v_k = \frac{u_k}{\|u_k\|}$, which, up to a subsequence, converges weakly to some $v_0 \in W$ and strongly in $L^p(\Omega) \cap L^p(\partial\Omega, \rho)$. From $I(u_k) = 1$ we infer that $I(v_0) = 0$. If $v_0 = 0$ then from

$$\int_\Omega |\nabla v_0|^p\, dx \leq \liminf_{k \to \infty} \left(\frac{E_V(u_k)}{\|u_k\|^p} - \int_\Omega V|v_k|^p\, dx \right) = 0$$

we deduce that $v_k \to v_0 = 0$ strongly in $W^{1,p}(\Omega)$, a contradiction with $\|v_k\| = 1$. Thus it must be $v_0 \neq 0$. From $(PSC1)$ we infer $E_V(v_0) \leq \liminf_{k \to \infty} \frac{E_V(u_k)}{\|u_k\|^p} = 0$, then $\frac{v_0}{\|v_0\|_p}$ realizes α. By Theorem 4.1(2)(b), v_0 is a definite eigenfunction for (P) associated to λ_*. If, say, $v_0 > 0$, then the sequence u_k^- converges in measure to 0. By Proposition 6.3 and (8.6) we conclude that the sequence u_k^- is bounded in W. Then, up to a subsequence, u_k^- converges weakly to some z_0 that satisfies $E_V(z_0) = \frac{1}{2}\lambda_*$ and $I(z_0) = \frac{1}{2}$. Hence $2^{\frac{1}{p}} z_0 \in \mathcal{M}^+$ realizes λ_*. Thus $z_0 > 0$, a contradiction with the fact that u_k^- converges to 0 in measure. Therefore the sequence u_k is bounded. Since the sequence u_k is bounded, passing to the limit for a weakly convergent subsequence, we prove that the value λ_* is achieved at some point of $\mathcal{M}^+$, a contradiction. We have just proved that $\mu > \lambda_*$.

(ii) Now, let us pick a function u_1 (for instance, a function belonging to the sequence defined in (4.3)) such that $E_V(u_1) < \mu$. That λ_2 defined in the statement is an eigenvalue for problem (P) is a consequence, for instance of the version of Mountain Pass Theorem on C^1-manifolds of [7, Theorem 4.1]. Observe that: (1) by Lemma 8.5, $(PSC)_{\lambda_2}$ holds because $\lambda_* < \mu \le \lambda_2$ and (2) the geometric condition

$$\max\{E_V(u_1), E_V(-u_1)\} < \mu \le \lambda_2$$

also holds.

(iii) Let us prove that $\lambda_2 \le \mu$. Let $\epsilon > 0$ be small enough and $h \in \mathcal{L}$ such that

$$\max_h E_V \le \mu + \epsilon.$$

Put $u_0 = h(0)$. We claim that there exists a path $\tilde{h}$ in $\mathcal{M}^+$ from u_0 to u_1 such that E_V stays below the level $\mu + \epsilon$ on $\tilde{h}$ and consequently $\lambda_2 \le \mu + \epsilon$ and the conclusion follows (since a similar argument holds for $h(1)$ and $-u_1$). For that purpose, we consider $V + t$ for $t > 0$ small in order to have $\alpha(V + t) > 0$ and consequently the Palais–Smale condition will be satisfied everywhere. We also choose $t > 0$ small enough to have $\max\{E_{V+t}(u_0), E_{V+t}(u_1)\} < \mu(V) + \epsilon$. Let us consider the open set $\mathcal{O} \stackrel{\text{def}}{=} \{u \in \mathcal{M}^+ \mid E_{V+t}(u) < \mu(V) + \epsilon\}$. Observe that, if $t > 0$ is small enough, then $\mu(V) < \mu(V + t)$. It follows from Lemma 14 of [6] that $\mathcal{O}$ has at most two arcwise connected components (because $\varphi_1(V + t)$ and $-\varphi_1(V + t)$ are the only critical points of the restriction of E_{V+t} to $\mathcal{M}^+$ in $\mathcal{O}$). If u_1 and $-u_1$ lie in the same component then it comes that $\lambda_2 \le \mu(V) + \epsilon$. Otherwise u_0 can be connected by a path to either u_1 or $-u_1$. But since $E_{V+t}(u) = E_{V+t}(\pm|u|)$ for every $u \in \mathcal{M}^+$, we can always find a path from u_0 to u_1 by taking absolute value.

(iv) Finally, the fact that $\lambda_2 = \mu$ is the first nonprincipal eigenvalue is due to the following observation. If $\lambda > \lambda_*$ is an eigenvalue for (P) and u is a corresponding eigenfunction, then the path h defined as $h(t) = \frac{tu^+ - (1-t)u^-}{I(tu^+ - (1-t)u^-)^{1/p}}$ is well defined (we leave the details to the reader) and belongs to $\mathcal{H}$. Moreover $E_V(h(t)) = \lambda$ for all $t \in [0, 1]$ because $E_V(u^\pm) = \lambda I(u^\pm)$. Thus it comes from the definition of μ that $\mu \le \lambda$. □

Here below we give an example of problem (P) in dimension $N = 1$ with a weight σ changing sign showing that one can have *only one* eigenvalue on the right of λ_1.

Example 8.7.1. Let us consider the problem

$$\begin{cases} -u'' + u = -\lambda u & \text{in } [0,1] \\ -\sigma_1 u'(0) = \lambda u(0), \\ \sigma_2 u'(1) = \lambda u(1) \end{cases}$$

with $\sigma_1 + \sigma_2 = -1$ and $\sigma_1 < -1$. Hence $\lambda_1 = -1$ is a principal eigenvalue with $u(t) = t + \sigma_1$ as corresponding eigenfunction. Any solution of the differential equation for $\beta^2 := \lambda + 1 > 0$ is of the form $u(x) = Ae^{\beta x} + Be^{-\beta x}$ and the boundary condition are satisfied if and only if

$$e^{2\beta} = \frac{\sigma_1\sigma_2\beta^2 - \beta(\beta^2 - 1) + (\beta^2 - 1)^2}{\sigma_1\sigma_2\beta^2 + \beta(\beta^2 - 1) + (\beta^2 - 1)^2} := h(\beta).$$

A simple analysis of h shows that $h(\beta) \le 1 + 2\beta$, and that equality holds if and only if $\beta = 0$. Thus, $\lambda = \lambda_1 = -1$ and there are not eigenvalues greater than $\lambda_1 = -1$.

Acknowledgments

This work was partially carried out while the first author was visiting the IMSP of the *Université d'Abomey Calavi (Porto-Novo)* and also while the second author was visiting the *Université du Littoral*

Côte d'Opale (ULCO) and the *Université Libre de Bruxelles (ULB)*. We would like to express our gratitude to those institutions.

Appendix A

We want to give here a proof of the boundedness of the solutions of problem (P). This type of result is well established for Dirichlet or Newmann boundary conditions and it has already been proved by several authors, see for instance [16]. However we didn't find any precise reference for the type of equations with the mixed boundary conditions that we are considering here. We include the proof of the following theorem for sake of completeness.

Theorem A.1. *Let Ω be a bounded regular domain satisfying the main assumptions of the introduction. Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \Gamma_1 \times \mathbb{R} \to \mathbb{R}$ be two Carathéodory functions satisfying the growth assumption*

$$\left|f(x,t)\right| \le a|t|^{q-1} + b \quad \forall (x,t) \in \Omega \times \mathbb{R}; \tag{A.1}$$

$$\left|g(x,t)\right| \le a|t|^{r-1} + b \quad \forall (x,t) \in \Gamma_1 \times \mathbb{R}, \tag{A.2}$$

for some $a, b \in \mathbb{R}^+$, $1 \le q \le p^*$ and $1 \le r \le p_*$. Then there exists a constant $C = C(a, b, q, r, \Omega, \Gamma_1, \|u\|_{p^*}, \|u\|_{p_*,\Gamma_1})$ such that any weak solution u of

$$\begin{cases} -\Delta_p u = f(x,u) & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = g(x,u) & \text{on } \Gamma_1, \\ u = 0 & \text{on } \Gamma_2, \end{cases} \tag{A.3}$$

satisfies

(i) if $q < p^*$ and $r < p_*$ then

$$\|u\|_\infty + \|u\|_{\infty,\Gamma_1} \le C,$$

(ii) if $q = p^*$ or $r = p_*$ then $u \in L^t(\Omega) \cap L^t(\partial\Omega, \rho)$ for all $t \in [1, +\infty[$ and there exists a constant C_t, depending on the previous parameters and on t, such that

$$\|u\|_t + \|u\|_{t,\Gamma_1} \le C_t.$$

Proof. By Sobolev's embedding theorem, it suffices to consider the case $N \ge p$. We may assume that $u \ge 0$ otherwise we will consider test functions involving u^+ and u^- to obtain the a-priori bounds.

(i) Let us assume that $1 \le q < p^*$ and $1 \le r < p_*$. For $M > 0$ we define $u_M(x) := \min\{u(x), M\}$ and $\Phi(x) := u_M^{kp+1}(x) \in W \cap L^\infty(\Omega) \cap L^\infty(\partial\Omega, \rho)$ for $k > 0$ to be determined later. Using Φ a test function in (A.3), one obtains

$$c_k^{-1}\left\|\nabla\left(u_M^{k+1}\right)\right\|_p^p \le a\left[\int_\Omega u_M^{kp+q}\,dx + \oint_{\Gamma_1} u_M^{kp+r}\,d\rho\right] + b\left[\int_\Omega u_M^{kp+1}\,dx + \oint_{\Gamma_1} u_M^{kp+1}\,d\rho\right]$$

where $c_k := \frac{(k+1)^p}{kp+1}$. From now on C will denote a generic constant independent of k, but depending on p, q, r, and Ω. Since $c_k > 1$ for any $k > 0$, adding $\|u_M^{(k+1)}\|_{p,\Gamma_1}^p$ to each side, letting $M \to +\infty$ and using Young's inequality twice it comes

$$\left\|u^{(k+1)}\right\|_W \le C c_k^{1/p}\left(\int_\Omega u^{kp+q}\,dx + \oint_{\Gamma_1} u^{kp+r}\,d\rho + 1\right)^{1/p}.$$

Since our purpose is to make $k \to +\infty$ we will replace all the constants of the form $C^{1/k}$ that should appear in Holder's estimates by a generic constant C.

By Sobolev's embedding theorem, there exists C_1 such that

$$\left\|u^{k+1}\right\|_{p^*} + \left\|u^{k+1}\right\|_{p_*,\Gamma_1} \le C_1\left\|u^{k+1}\right\|_W$$

and hence

$$\|u\|_{p^*(k+1)} + \|u\|_{p_*(k+1),\Gamma_1} \le C c_k^{1/p(k+1)}\left(\int_\Omega u^{kp+q}\,dx + \oint_{\Gamma_1} u^{kp+r}\,d\rho + 1\right)^{1/p(k+1)}. \tag{A.4}$$

Put $k_1 = \min\{\frac{p^*-q}{p}, \frac{p_*-r}{p}\}$, $\xi_0 = \|u\|_{p^*} + \|u\|_{p_*}$ and $\xi_1 = \|u\|_{p^*(k_1+1)} + \|u\|_{p_*(k_1+1),\Gamma_1}$. Using Holder's inequality and that $\max\{k_1 p + q, k_1 p + r\} \le p^*$ it comes from (A.4) that

$$\xi_1 \le C c_{k_1}^{1/p(k_1+1)}(\xi_0 + 1)^{\frac{p^*}{p(k_1+1)}}.$$

Define successively

$$k_n = \min\left\{\frac{(k_{n-1}+1)p^* - q}{p}, \frac{(k_{n-1}+1)p_* - r}{p}\right\}, \qquad c_n := c_{k_n}.$$

Observe that $k_n > 0$ for all $n \in \mathbb{N}$. Let us denote for simplicity $\xi_n := \|u\|_{p^*(k_n+1)} + \|u\|_{p_*(k_n+1),\Gamma_1}$. By induction we get from (A.4) that

$$\xi_n \le C c_n^{1/p(k_n+1)}(\xi_{n-1} + 1)^{p^*/p(k_n+1)}. \tag{A.5}$$

Observe that $\lim_{n\to+\infty} k_n = +\infty$ and therefore $u \in L^r(\Omega) \cap L^r(\partial\Omega, \rho)$ for all $r > 1$. To obtain a uniform bound for u, we define a new sequence

$$q_0 := p_*, \qquad q_{n+1} := p_*\left(\frac{q_n}{sp} + \frac{1}{p'}\right)$$

where s is any fixed number satisfying $1 < s < \frac{p_*}{p}$. Let us show the new estimate

$$\left(\|u\|_{q_{n+1}} + \|u\|_{q_{n+1},\Gamma_1}\right)^{q_{n+1}} \le C\left(\frac{q_{n+1}^p}{q_n}\right)^{p_*/p}\left(\|u\|_{q_n} + \|u\|_{q_n,\Gamma_1}\right)^{q_n p_*/sp} \tag{A.6}$$

Indeed, since we know that $u \in L^r(\Omega) \cap L^r(\partial\Omega, \rho)$ for all $r > 1$, we can use $u^{q_n/s}$ as a test function to get from the right hand side of Eq. (A.3)

$$\begin{aligned}
R &= \int_\Omega f(x,u)u^{q_n/s}\,dx + \oint_{\Gamma_1} g(x,u)u^{q_n/s}\,d\rho \\
&\le \int_\Omega \left[au^{q-1} + b\right]u^{q_n/s}\,dx + \oint_{\Gamma_1}\left[au^{r-1} + b\right]u^{q_n/s}\,d\rho \\
&\le D\left(\|u\|_{q_n} + \|u\|_{q_n,\Gamma_1}\right)^{q_n/s}
\end{aligned}$$

with $D = \|au^{q-1} + b\|_{s'} + \|au^{r-1} + b\|_{s',\Gamma_1}$, and in the left hand side (the gradient term) of (3.8), after using Sobolev's embedding as above,

$$C_1 \frac{q_n}{s}\left(\frac{q_n}{sp} + \frac{1}{p'}\right)^{-p}\left(\|u\|_{q_{n+1}} + \|u\|_{q_{n+1},\Gamma_1}\right)^{q_{n+1}\frac{p}{p_*}} \le L,$$

where

$$L = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla\left(u^{q_n/s}\right) dx.$$

The equality $R = L$ gives the estimate (A.6). For simplicity we denote

$$\theta_n := q_n \ln\left(\|u\|_{q_n} + \|u\|_{q_n,\Gamma_1}\right), \qquad B_n := \ln\left(C\left(\frac{q_{n+1}^p}{q_n}\right)^{p_*}\right).$$

By induction we get from (A.6) that

$$\theta_{n+1} \le B_n + (p_*/sp)\theta_n.$$

Then

$$\theta_n \le (p_*/sp)^n\theta_0 + \sum_{i=1}^n (p_*/sp)^i B_{n-i}. \tag{A.7}$$

A simple estimate gives

$$d_0 \le \sum_{i=1}^n (p_*/sp)^i B_{n-i} \le d_1(p_*/sp)^n$$

for some $d_0, d_1 > 0$, and also we have $q_n \ge d_2(p_*/sp)^n$ for some $d_2 > 0$. Then from (A.7)

$$\frac{\theta_n}{q_n} \le \frac{(\theta_0 + d_1)(p_*/sp)^n}{q_n} \le \frac{\theta_0 + d_1}{d_2}$$

and hence

$$\|u\|_{q_n} + \|u\|_{q_n,\Gamma_1} \le e^{\frac{\theta_0 + d_1}{d_2}}.$$

We conclude by letting $n \to +\infty$.

(ii) Assume for instance that $q = p^*$ and $r = p_*$. The proof in the case when one of these exponents is subcritical can be done using both the arguments of case (i) and those of the present case. The details are left to the reader. Assume as before that $u \ge 0$ and take, for any $t \ge p$, the test function $\Phi(x) = u_M^{tp+1}(x)$. After multiplying Eq. (A.3) by Φ, integrating by parts and letting $M \to +\infty$, one obtains, from the gradient term, in the left hand side,

$$L := \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\Phi\, dx = C_t \int_\Omega |\nabla u^{\frac{t}{p}}|^p\, dx \ge S_t\left(\|u\|^t_{\frac{tp^*}{p}} + \|u\|^t_{\frac{tp_*}{p},\Gamma_1}\right),$$

where we have used Sobolev's embedding to obtain the last inequality. From the right hand side we get

$$R := \int_{\Omega} f(x,u)\Phi\,dx + \int_{\Gamma_1} g(x,u)\Phi\,d\rho$$
$$\leq a\left[\int_{\Omega} u^{p^*+t-p}\,dx + \oint_{\Gamma_1} u^{p_*+t-p}\,d\rho\right] + b\left[\int_{\Omega} u^{t-p+1}\,dx + \oint_{\Gamma_1} u^{t-p+1}\,d\rho\right].$$

To estimate the 1st and the 2nd integrals of R we define, for all $m > 0$, the sets

$$\Omega_m := \{x \in \Omega \mid u(x) \geq m\}, \qquad \Gamma_m := \{x \in \Gamma_1 \mid u(x) \geq m\}$$

and we obtain, using again Holder's inequality,

$$\int_{\Omega} u^{p^*+t-p}\,dx \leq m^{p^*-p} \int_{\Omega\setminus\Omega_m} u^t\,dx + \int_{\Omega_m} u^{p^*+t-p}\,dx$$
$$\leq m^{p^*-p}\|u\|_t^t + \|u\|^t_{\frac{tp^*}{p}}\left(\int_{\Omega_m} u^{p^*}\,dx\right)^{p/N};$$

and similarly

$$\oint_{\Gamma_1} u^{p_*+t-p}\,d\rho \leq m^{p_*-p}\|u\|^t_{t,\Gamma_1} + \|u\|^t_{\frac{tp_*}{p},\Gamma_1}\left(\oint_{\Gamma_m} u^{p_*}\,d\rho\right)^{(p-1)/(N-1)}.$$

Choose $m > 0$ such that $(\int_{\Omega_m} u^{p^*}\,dx)^{p/N} + (\oint_{\Gamma_m} u^{p_*}\,d\rho)^{(p-1)/(N-1)} < S_t/2$ and use that $R = L$ to get

$$S_t/2\left(\|u\|^t_{\frac{tp^*}{p}} + \|u\|^t_{\frac{tp_*}{p},\Gamma_1}\right) \leq C_t\left(\|u\|_t^t + \|u\|^t_{t,\Gamma_1} + \int_{\Omega} u^{t-p+1}\,dx + \oint_{\Gamma_1} u^{t-p+1}\,d\rho\right)$$

and, after using Holder's and Young's inequalities for the 3rd and 4th integrals,

$$\frac{S_t}{4}\left(\|u\|^t_{\frac{tp^*}{p}} + \|u\|^t_{\frac{tp_*}{p},\Gamma_1}\right) \leq C_t\left(\|u\|_t^t + \|u\|^t_{t,\Gamma_1} + 1\right). \tag{A.8}$$

Finally, we define

$$t_0 := p, \qquad t_n := p\left(\frac{p_*}{p}\right)^n$$

to obtain from (A.8) successively that $u \in L^{\frac{t_n p^*}{p}}(\Omega) \cap L^{\frac{t_n p_*}{p}}(\partial\Omega, \rho)$. Since $t_n \to +\infty$, we get the conclusion. $\square$

References

[1] J. Adams, Sobolev Spaces, Academic Press, New York, 1975.
[2] G.A. Afrouzi, K.J. Brown, On principal eigenvalues for boundary eigenvalue problems with indefinite weight and Robin boundary conditions, Proc. Amer. Math. Soc. 127 (1) (1999) 125–130.
[3] W. Allegretto, Y.-X. Huang, A Picone's identity for the p-laplacian and applications, Nonlinear Anal. 32 (7) (1998) 819–830.

[4] W. Allegretto, A. Mingarelli, On the nonexistence of positive solutions for a Schrödinger equation with an indefinite weight-function, C. R. Math. Rep. Acad. Sci. Can. 8 (1) (1986) 69–73.
[5] A. Anane, N. Tsouli, On the second eigenvalue of the p-Laplacian, in: Nonlinear Partial Differential Equations, Fès, 1994, in: Pitman Res. Notes Math. Ser., vol. 343, Longman, Harlow, 1996, pp. 1–9.
[6] M. Arias, J. Campos, M. Cuesta, J.-P. Gossez, Asymmetric elliptic problems with indefinite weights, Ann. Inst. H. Poincare Anal. Non Lineaire 19 (2002) 581–616.
[7] M. Arias, J. Campos, M. Cuesta, J.-P. Gossez, An asymmetric Neumann problem with weights, Ann. Inst. H. Poincare Anal. Non Lineaire 25 (2) (2008) 267–280.
[8] C. Bandle, W. Reichel, A linear parabolic problem with non-dissipative dynamical boundary conditions, in: M. Chipot, H. Ninomiya (Eds.), Recent Advances on Elliptic and Parabolic Issues, Proceedings of the 2004 Swiss–Japanese Seminar, World Scientific, 2006, pp. 46–79.
[9] C. Bandle, J. von Below, W. Reichel, Parabolic problems with dynamical boundary conditions: eigenvalue expansions and blow up, Rend. Lincei Math. Appl. 17 (2006) 35–67.
[10] P.A. Binding, Y.X. Huang, The eigencurve for the p-laplacian, Differential Integral Equations 8 (2) (1995) 405–414.
[11] P.A. Binding, Y.X. Huang, Existence and nonexistence of positive eigenfunctions for p-Laplacian, Proc. Amer. Math. Soc. 123 (6) (1995) 1833–1838.
[12] M. Cuesta, Eigenvalue problems for the p-Laplacian with indefinite weights, Electron. J. Differential Equations 2001 (33) (2001) 1–9.
[13] M. Cuesta, H. Ramos Quoirin, A weighted eigenvalue problem for the p-laplacian plus a potential, NoDEA Nonlinear Differential Equations Appl. 16 (4) (2009) 469–491.
[14] E. DiBenedetto, $C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations, Nonlinear Anal. 7 (8) (1983) 827–850.
[15] J. Fleckinger, J. Hernandez, F. de Thélin, Existence of multiple principal eigenvalues for some indefinite linear eigenvalue problems, Boll. Unione Mat. Ital. (8) 7-B (2004) 159–188.
[16] M. Guedda, L. Véron, Quasilinear elliptic equations involving critical Sobolev exponents, Nonlinear Anal. 13 (8) (1989) 879–902.
[17] L. Leadi, A. Marcos, A weighted eigencurve for Steklov problems with a potential, NoDEA Nonlinear Differential Equations Appl. 16 (2013) 687–713, http://dx.doi.org/10.1007/s00030-012-0175-0.
[18] G. Lieberman, Boundary regularity for solutions of degenerate elliptic equations, Nonlinear Anal. 12 (11) (1988) 1203–1219.
[19] P. Lindqvist, On the equation $div(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$, Proc. Amer. Math. Soc. 109 (1990) 157–164; Addendum, Proc. Amer. Math. Soc. 116 (1992) 583–584.
[20] J. Lopez-Gomez, The maximum principle and the existence of principal eigenvalues for some linear weighted eigenvalue problems, J. Differential Equations 127 (1996) 263–294.
[21] K. Perera, R.P. Agarwal, D. O'Regan, Morse Theoretic Aspects of p-Laplacian Type Operators, Math. Surveys Monogr., vol. 161, A.M.S, 2010.
[22] A. Szulkin, Ljusternik–Schnirelmann theory on C^1 manifolds, Ann. Inst. H. Poincare Anal. Non Lineaire (1988) 119–139.
[23] K. Umezu, On eigenvalue problems with Robin type boundary conditions having indefinite coefficients, Appl. Anal. 85 (2006) 1313–1325.
[24] J.L. Vázquez, A strong maximum principle for some quasilinear elliptic equations, Appl. Math. Optim. 12 (1984) 191–202.