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THE GROWTH FUNCTION OF THE VOLUME OF GEODESIC BALLS IN RIEMANNIAN MANIFOLDS OF HYPERBOLIC TYPE

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Abstract. Let (M, g) be a compact Riemannian manifold of hyperbolic type and X be its universal Riemannian covering. We study in this paper, the growth function of the geodesic balls of X. We show that the critical exponent of the group of deck transformations of X is equal to the volume entropy h_g of M.

1. Introduction

A compact Riemannian manifold (M, g) is called of hyperbolic type if there exists an another Riemannian metric g_0 such that (M, g_0) has a strictly negative curvature.

Note that, in dimension 2, an orientable manifold M is of hyperbolic type if and only if its genus is ≥ 2.

We say that a function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is of purely exponential type if there exist constants $a > 1$ and $r_0 > 0$ such that

$$\frac{1}{a} \leq \frac{f(r)}{e^{hr}} \leq a \qquad \forall r \geq r_0,$$

for some constant $h > 0$. The real number h is called the exponential factor of f.

In 1969, Margulis proved, for suitable constant $h > 0$, the existence of

$$a(p) := \lim_{r\to\infty} \frac{vol\ S(p, r)}{e^{hr}}$$

at each point p in manifolds of strictly negative curvature and that the function a is continuous (see [21]). Clearly, this result implies purely exponential growth of volume of geodesic spheres.

If (M, g) is a compact Riemannian manifold, Manning has introduced an interesting asymptotic invariant (volume entropy) h_g given as follows : if $vol\ B_g(p, r)$ denotes the volume of the geodesic ball $B_g(p, r)$ with centre p and radius r in the universal Riemannian covering X of (M, g), then we have

$$h_g := \lim_{r\to\infty} \frac{\log\ vol\ B_g(p, r)}{r},$$

where the limit on the right hand side exists for all $p \in X$ and, in fact, is independent of p. Manning showed that, in the case of non positive curvature, h_g coincides with the topological entropy (see [20]).

In 1997, using the notions of Busemann density and Patterson Sullivan measure, G. Knieper proved the following result (see [19]) :

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If (M, g_0) is a compact rank-1 Riemannian manifold of non-positive curvature and X_0 its universal Riemannian covering, there exist constants $a_0 \geq 1$ and $r_0 \geq 0$ such that

$$\frac{1}{a_0} \leq \frac{vol\ S_{g_0}(p,r)}{e^{h_{g_0} r}} \leq a_0 \quad \forall r \geq r_0,$$

where h_{g_0} is the volume entropy of (M, g_0) and $S_{g_0}(p,r)$ is the geodesic sphere in X_0 with centre p and radius r.

The main result of this paper is :

Theorem 1.1. *Let (M, g) be a compact Riemannian manifold of hyperbolic type and X be its universal Riemannian covering. Then the growth function of the volume of geodesic balls of X is of purely exponential type with the volume entropy h_g as exponential factor.*

Remark 1.1. *Note that the manifolds considered in Theorem 1.1 may have curvature of both signs (see ([8], p.152) or ([15], p.199)). This result yields a sufficient condition for the non existence of Riemannian metric with strictly negative curvature on a compact manifold.*

The paper is organized as follows : In section 2 we study the ideal boundary and the Gromov boundary of a manifold of hyperbolic type. In section 3 we introduce a notion of quasi-convex cocompact group which we use to prove Theorem 1.1.

2. Gromov and ideal boundaries of manifolds of hyperbolic type

Let recall first some basic notions about a compactification of Hadamard manifolds.

Definition 2.1. *A connected, simply-connected and complete Riemannian manifold is called Hadamard manifold.*

Let (X_0, g_0) be a Hadamard manifold. Two geodesics $c_1, c_2 : \mathbb{R} \to X_0$ are said to be asymptotic, if there exists aconstant $D \geq 0$ such that

$$d_{g_0}(c_1(t), c_2(t)) < D \quad \forall t \geq 0.$$

This defines an equivalence relation on the set of geodesics of X_0.

An equivalence class of this relation is called point at infinity of X_0. If $c : \mathbb{R} \to X_0$ is a geodesic, its equivalence class is denoted by $c(+\infty)$. Let $c^{-1} : \mathbb{R} \to X_0$ define by $c^{-1}(t) := c(-t) \quad \forall t \in \mathbb{R}$. The equivalence class of c^{-1} is denoted by $c(-\infty)$.

The ideal boundary $X_0(\infty)$ of X_0 is the set of equivalence classes of the geodesics of X_0.

One define a natural topology on the set $\overline{X}_0 := X_0 \cup X_0(\infty)$ as follows:

Let consider the set $B(x,1) = \{v \in T_x X_0 \mid \|v\| \leq 1\}$ and the bijection

$$\Phi_x : B(x,1) \longrightarrow \overline{X}_0 = X_0 \cup X_0(\infty)$$
$$v \longmapsto \begin{cases} \exp_x(\frac{\|v\|}{1-\|v\|})v, & \text{si } \|v\| < 1 \\ c_v(+\infty) & \text{si } \|v\| = 1 \end{cases},$$

where c_v is the geodesic satisfying $c_v(0) = x$ and $\dot{c}_v(0) = v$. We have the following Lemma.

Lemma 2.0.1. *Let (X_0, g_0) be a Hadamard manifold, $x \in X_0$ and $\xi \in X_0(\infty)$. Then there exists a unique geodesic $c : \mathbb{R} \to X_0$ satisfying $c(0) = x$ and $c(+\infty) = \xi$.*

Proof. (see [2] or [8]). □

For $p \in X_0$, q_1 and $q_2 \in \overline{X}_0 = X_0 \cup X_0(\infty)$ with $p \neq q_1$ and $p \neq q_2$, we define

$$\angle_p(q_1, q_2) := \angle(\dot{c}_{pq_1}(0), \dot{c}_{pq_2}(0)),$$

where $c_{pq_i} : \mathbb{R} \to X_0$ is the geodesic joining the points p and q_i if $q_i \in X_0$ and $c_{pq_i}(0) = p$ and $c_{pq_i}(\infty) = q_i$ if $q_i \in X_0(\infty)$ and $\angle(\dot{c}_{pq_1}(0), \dot{c}_{pq_2}(0))$ is the angle subtended by the vectors $\dot{c}_{pq_1}(0)$ and $\dot{c}_{pq_2}(0)$.

For $p \in X_0$, $\xi \in X_0(\infty)$, $\epsilon > 0$ and $R = 0$, let

$$\Gamma_p(\xi, \epsilon, R) := \{q \in \overline{X}_0 = X_0 \cup X_0(\infty) \mid q \neq p, \angle_p(q, \xi) < \epsilon \text{ and } d_{g_0}(p, q) > R\}$$

For a fixed point $p \in X_0$, the set of all $\Gamma_p(\xi, \epsilon, R)$ and the open sets of X_0 generate a topology on $\overline{X}_0 = X_0 \cup X_0(\infty)$. This topology is called the cône topology. With respect to this topology, the set $\overline{X}_0 := X_0 \cup X_0(\infty)$ is homeomorphic to a closed n-ball in $\mathbb{R}^n$ (see [2] or [8]). The induced topology on $X_0(\infty)$ is called the sphere topology.

Definition 2.2. *Let (X_1, d_1) and (X_2, d_2) be two metric spaces. A map $\phi : X_1 \longrightarrow X_2$ is called a $(A, \alpha)-$quasi-isometric map, for some constants $A > 1$ and $\alpha > 0$ if :*

$$\frac{1}{A} d_1(x, y) - \alpha \leq d_2(\phi(x), \phi(y)) \leq A d_1(x, y) + \alpha \quad \forall x,\ y \in X_1.$$

In a metric space X, a $(A, \alpha)-$quasi-geodesic (resp. $(A, \alpha)-$quasi-geodesic ray) is a $(A, \alpha)-$quasi-isometric map $\phi : \mathbb{R} \longrightarrow X$ (resp. $\phi : \mathbb{R}^+ \longrightarrow X$).

Definition 2.3. *Let (X, d) be a metric space, E and F subsets of X. The Hausdorff distance d_H is defined by :*

$$d_H(E, F) := \inf \left\{ r > 0 \ / \ E \subset T_r(F) \text{ and } F \subset T_r(E) \right\}$$

where

$$T_r(G) := \{x \in X \ / \ d(x, G) \leq r\}. \qquad \forall G \subset X$$

Theorem 2.1. *(Morse Lemma)*

Let (X_0, g_0) be a Hadamard manifold with sectional curvature $K_{X_0} \leq -k_0^2 < 0$ for some constant $k_0 > 0$. Then for each $(A, \alpha)-$quasi-geodesic (resp. $(A, \alpha)-$quasi-geodesic ray) $\phi : \mathbb{R} \longrightarrow X_0$ (resp. $\phi : \mathbb{R}^+ \longrightarrow X_0$), there exists a geodesic (resp. geodesic ray) $c : \mathbb{R} \longrightarrow X_0$ (resp. $c : \mathbb{R}^+ \longrightarrow X_0$) such that $d_H(c(\mathbb{R}), \phi(\mathbb{R})) \leq r_0$ (resp. $d_H(c(\mathbb{R}^+), \phi(\mathbb{R}^+)) \leq r_0$); r_0 depends only on A, α and k_0.

Proof. (see [16]) □

Definition 2.4. *Let (X, d) be a metric space with a reference point x_0. The Gromov product of the points x and y of X with respect to x_0 is the nonnegative real number $(x \cdot y)_{x_0}$ defined by :*

$$(x \cdot y)_{x_0} = \frac{1}{2}\{d(x, x_0) + d(y, x_0) - d(x, y)\}.$$

A metric space (X, d) is said to be a δ-hyperbolic space for some constant $\delta \geq 0$, if

$$(x \cdot y)_{x_0} \geq \min\{(x \cdot z)_{x_0}; (y \cdot z)_{x_0}\} - \delta$$

for all x, y, z and every choice of reference point x_0. We call X a Gromov hyperbolic space if it is a δ-hyperbolic space for some $\delta \geq 0$. The usual hyperbolic space $\mathbb{H}^n$ is a $\delta-$hyperbolic space, where $\delta = \log 3$. More generally, every Hadamard manifold with sectional curvature $\leq -k^2$ for some constant $k > 0$ is a δ-hyperbolic space, where $\delta = k^{-1} \log 3$ (see [1], [5], [12] *or* [13]*).*

Lemma 2.1.1. *Let (X, d) be a complete geodesic δ-hyperbolic space, x_0 a reference point in X, x and y two points of X. Then*

$$d(x, \gamma_{xy}) - 4\delta \leq (x \cdot y)_{x_0} \leq d(x, \gamma_{xy})$$

for every geodesic segment γ_{xy} joining x and y.

Proof. (see [5] or [6]). □

Now let X be a Gromov hyperbolic manifold, x_0 a reference point in X. We say that the sequence $(x_i)_{i\in\mathbb{N}}$ of points in X converges at infinity if

$$\lim_{i,j\to\infty} (x_i \cdot x_j)_{x_0} = \infty.$$

If x_1 is another reference point in X,

$$(x \cdot y)_{x_0} - d(x_0, x_1) \leq (x \cdot y)_{x_1} \leq (x \cdot y)_{x_0} + d(x_0, x_1).$$

Then the definition of the sequence that converges at infinity depends not on the choice of the reference point. Let recall the following equivalence relation $\mathcal{R}$ on the set of sequences of points in X that converge at infinity :

$$(x_i)\mathcal{R}(y_j) \Longleftrightarrow \lim_{i,j\to\infty} (x_i \cdot y_j)_{x_0} = \infty.$$

The Gromov boundary $X^G(\infty)$ of X is the set of the equivalence classes of sequences that converge at infinity.

Let X be a simply connected Riemannian manifold which is a Gromov hyperbolic space. One defines on the set $X\cup X^G(\infty)$ a topology as follows (see [5] page 22 or [12] page 122) :

(1) if $x \in X$, a sequence $(x_i)_{i\in\mathbb{N}}$ converges to x with respect to the topology of X.
(2) if $(x_i)_{i\in\mathbb{N}}$ defines a point $\xi \in X^G(\infty)$,$(x_i)_{i\in\mathbb{N}}$ converges to ξ.
(3) For $\eta \in X^G(\infty)$ and $k > 0$, let

$$V_k(\eta) := \left\{ y \in X \cup X^G(\infty) \;/\; (y \cdot \eta)_{x_0} > k \right\},$$

where

$$(x \cdot y)_{x_0} = \inf \left\{ \liminf_{i\to\infty} (x_i \cdot y_i)_{x_0} \;/\; x_i \to x,\; y_i \to y \right\}$$

for x and y elements of $X\cup X^G(\infty)$.

The set of all $V_k(\eta)$ and the open metric balls of X generate a topology on $X\cup X^G(\infty)$. With respect to this topology, X is dense in $X\cup X^G(\infty)$ and $X\cup X^G(\infty)$ is compact.

Lemma 2.1.2. *(see* [6]*) Let X be a δ-hyperbolic space. Then*

(1) *Each geodesic $\gamma : \mathbb{R} \longrightarrow X$ defines two distinct points $\gamma(+\infty)$ and $\gamma(-\infty)$.*
(2) *For each $(\eta, x) \in X^G(\infty)\times X$, there exists a geodesic ray γ such that $\gamma(0) = x$ and $\gamma(+\infty) = \eta$. For any other geodesic ray γ' with $\gamma'(0) = \gamma(0) = x$ and $\gamma(+\infty) = \gamma'(+\infty) = \eta$ we have $d(\gamma'(t), \gamma(t)) \leq 4\delta$ for all $t \geq 0$.*

Definition 2.5. *Let $\xi \in X^G(\infty)$ and $c : \mathbb{R}_+ \longrightarrow X$ be a minimal geodesic ray satisfying $c(+\infty) = \xi$. The function*

$$b_c(x) := \lim_{t\to\infty} (d(x, c(t)) - t)$$

is well defined on X and is called the Busemann function for the geodesic c.

Lemma 2.1.3. *(see [6]) Let X be a $\delta-$hyperbolic space, $\xi \in X^G(\infty)$, $x,\ y \in X$ and c a geodesic ray with $c(0) = x$ and $c(+\infty) = \xi$. Then there exists a neighbourhood $\mathcal{V}$ of ξin $X \cup X^G(\infty)$ such that*

$$|b_c(y) - (d(z, y) - d(z, x))| \leq K \text{ for all } z \in \mathcal{V} \cap X,$$

where b_c is the busemann function for the geodesic c and K is a constant depending only on δ.

Lemma 2.1.4. *Let X_1 be a metric space and (X_2, d_2) be a geodesic Gromov hyperbolic space. If there exists a quasi-isometric map $\phi : X_1 \longrightarrow X_2$, then X_1 is also a Gromov hyperbolic space. Moreover, if the map*

$$x \longmapsto d_2(x, \phi(X_1))$$

is bounded above, $X_1^G(\infty) \simeq X_2^G(\infty)$.

Proof. (see [5]) . ☐

Now let (M, g) be a compact Riemannian manifold of hyperbolic type and X be its universal Riemannian covering. Let g_0 denotes an associated metric of strictly negative curvature on M. The universal Riemannian covering X_0 of (M, g_0) is a Hadamard manifold satisfying $K_{X_0} \leq -k_0^2 < 0$ for some constant $k_0 > 0$. Then X_0 and X are Gromov hyperbolic spaces. Moreover, $X^G(\infty) \simeq X_0^G(\infty)$.

Two geodesic rays c and c' are said to be asymptotic if there exists a constant $D \geq 0$ such that $d_H(c(\mathbb{R}_+), c'(\mathbb{R}_+)) \leq D$. This defines an equivalence relation on the set of minimizing $g-$geodesic rays of X. Let $X(\infty)$ be the set of equivalence classes of asymptotic minimizing $g-$geodesic rays. For each minimizing $g-$geodesic ray c of X, it follows from Morse Lemma that there exists a g_0-geodesic ray c_0 such that $d_H(c(\mathbb{R}_+), c_0(\mathbb{R}_+)) \leq r_0$, where r_0 is the constant in Morse Lemma. Let $[c]$ be the equivalence classe of minimizing $g-$geodesic ray c and $[c_0]$ be the equivalence classe of the g_0-geodesic c_0. The map f defines by :

$$\begin{array}{rccc} f : & X(\infty) & \longrightarrow & X_0(\infty) \\ & [c] & \longmapsto & [c_0] \end{array}$$

is bijective. Then f defines on $X(\infty)$ a natural topology with respect to which $X(\infty)$ and $X_0(\infty)$ are homeomorphic i.e. $X(\infty) \simeq X_0(\infty)$ (see [9]).

Lemma 2.1.5. *Let X_0 be a Hadamard manifold with sectional curvature $K_{X_0} \leq -k_0^2 < 0$ for some constant $k_0 > 0$. There exists a natural homeomorphism*

$$\phi : X_0 \cup X_0^G(\infty) \longrightarrow X_0 \cup X_0(\infty).$$

In particular, $X_0^G(\infty) \simeq X_0(\infty)$.

Proof. (see [4]). ☐

Using Morse lemma, Theorem 2.1. and the properties of the ideal boundaries, we obtain the following lemma :

Lemma 2.1.6. *Let (M, g) be a compact Riemannian manifold of hyperbolic type and X be its universal Riemannian covering. Let g_0 be an associated metric of strictly negative curvature on M and X_0 be the universal Riemannian covering of (M, g_0). We have*

$$X(\infty) \simeq X_0(\infty) \simeq X_0^G(\infty) \simeq X^G(\infty).$$

3. The growth rate of volume of balls in manifolds of hyperbolic type

Definition 3.1. *Let (X, d) be a gromov hyperbolic manifold with reference point x_0 and Γ be a discrete and infinite subgroup of the isometry group $Iso(X)$of X. For a given point $x \in X$, the limit set $\Lambda^g(\Gamma, x)$ is the set of the accumulation points of the orbit Γx in $X^G(\infty)$.*

Let (X, d) be a metric space and Γ be a discrete and infinite subgroup of the isometry group $Iso(X)$ of X. For $x_0,\ x \in X$ and $s \in \mathbb{R}$,

$$P_s(x, x_0) := \sum_{\gamma \in \Gamma} e^{-sd(x,\gamma x_0)}$$

denotes the Poincaré series associated to Γ. The number

$$\alpha := \inf \{s \in \mathbb{R}/P_s(x, x_0) < \infty\}$$

is called the critical exponent of Γ and is independent of x and x_0. The group Γ is called of divergence type if the Poincaré series diverges for $s = \alpha$. The following lemma introduces a usefull modification (due to Patterson) of the Poincaré series if Γ is not of divergence type.

Lemma 3.0.7. *Let Γ be a discrete group with critical exponent α. There exists a function $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$which is continuous, nondecreasing and such that*

$$for\ \ all\ \ a > 0,\ \ \lim_{r \to +\infty} \frac{f(r + a)}{f(r)} = 1$$

and the modified series

$$\tilde{P}_s(x, x_0) := \sum_{\gamma \in \Gamma} f(d(x, \gamma x_0))e^{-sd(x,\gamma x_0)}$$

converges for $s > \alpha$ and diverges for $s \leq \alpha$.

Proof. (see [23]). □

Now let (M, g) be a compact Riemannian manifold of hyperbolic type and X be its universal Riemannian covering. Let g_0 denote a metric of strictly negative curvature on M. The universal Riemannian covering X_0 of (M, g_0) is a Hadamard manifold satisfying $K_{X_0} \leq -k_0^2 < 0$ for some constant $k_0 > 0$. Let Γ be the group of deck transformations of X and α^{g_0} be its critical exponent with respect to the metric g_0. It follows from theorem 5.1 in [19] that :

$$\alpha^{g_0} = h_{g_0} := \lim_{r \to \infty} \frac{\log\ vol\ B_{g_0}(p, r)}{r}.$$

The fact that M is compact implies the existence of a constant $\lambda \geq 1$ such that the critical exponent α^g of Γ with respect to the metric g belongs to $[\lambda^{-1}h_{g_0}, \lambda h_{g_0}] \subset \mathbb{R}_+^*$ (see [18]).

Lemma 3.0.8. *Let (M, g) be a compact Riemannian manifold of hyperbolic type, X be its universal Riemannian covering and Γ be the group of deck transformations of X. Then :*

(1) $\Lambda^g(\Gamma, x) = \overline{\Gamma x} \cap X^G(\infty)$.
(2) $\gamma(\Lambda^g(\Gamma, x)) = \Lambda^g(\Gamma, x)$ *for all* $\gamma \in \Gamma$ *and* $x \in X$.
(3) $\Lambda^g(\Gamma, x)$ *is independent of* x.
(4) $\Lambda^g(\Gamma, x) = X^G(\infty)$

Proof. Using the definition of $\Lambda^g(\Gamma, x)$, we can easily check the properties (1) and (2).

3. For all $\xi \in \Lambda^g(\Gamma, x)$, by defnition there is a sequence $(\gamma_n)_n$ of points of Γ such that $\lim_{n\to\infty} \gamma_n x = \xi$. Then :

$$\lim_{m,n\to\infty} (\gamma_n x \cdot \gamma_m x)_{x_0} = \lim_{m,n\to\infty} [d(\gamma_n x, x_0) + d(\gamma_m x, x_0) - d(\gamma_n x, \gamma_m x)] = +\infty.$$

For all $y \in X$, we have :

$$\begin{array}{rcl} 2(\gamma_n x \cdot \gamma_n y)_{x_0} & = & d(\gamma_n x, x_0) + d(\gamma_n y, x_0) - d(\gamma_n x, \gamma_n y) \\ & \geq & d(\gamma_n x, x_0) + d(\gamma_n y, x_0) - d(x, y) \\ & \geq & d(\gamma_n x, x_0) - d(x, y). \end{array}$$

and

$$\begin{array}{rcl} d(\gamma_n x, x_0) & \leq & d(\gamma_n x, x) + d(\gamma_n y, y) + d(\gamma_n y, x_0) \\ & \leq & d(\gamma_n x, x_0) - d(x, y). \end{array}$$

Hence,

$$\lim_{n\to\infty} (\gamma_n x \cdot \gamma_n y)_{x_0} = +\infty \text{ and } \lim_{n\to\infty} \gamma_n y = \xi.$$

4. Let g_0 denote a metric of strictly negative curvature on M. The universal Riemannian covering X_0 of (M, g_0) is a Hadamard manifold satisfying $K_{X_0} \leq -k_0^2 < 0$ for some constant $k_0 > 0$. Then $\Lambda^{g_0}(\Gamma, x) = X_0(\infty)$ (see [18]). Finally, using lemma 2.1.6 we obtain that $\Lambda^g(\Gamma, x) = X^G(\infty)$. □

Let (X, g) be a Gromov hyperbolic manifold, Γ a non trivial subgroup of $Iso(X)$ and the limit set $\Lambda^g(\Gamma, x)$ of the orbit Γx in $X^G(\infty)$.

The Gromov hull $E(\Lambda^g(\Gamma, x))$ of $\Lambda^g(\Gamma, x)$ is the subset of X defined by the collection of the images of the goedesics $c : \mathbb{R} \longrightarrow X$ satisfying $c(-\infty) \in \Lambda^g(\Gamma, x)$ and $c(+\infty) \in \Lambda^g(\Gamma, x)$.

Definition 3.2. *A non trivial subgroup Γ of the isometry group $Iso(X)$ is quasi-convex cocompact if $E(\Lambda^g(\Gamma, x))/\Gamma$ is compact.*

The following lemma is due to Coornaert (see [6]).

Lemma 3.0.9. *Let (X, g) be a Gromov hyperbolic manifold with reference point x_0, Γ be a quasi-convex cocompact subgroup of the isometry group $Iso(X)$ with finite critical exponent α^g. Then, for all $x \in X$, there exists a constant $C_x \geq 1$ such that :*

$$\frac{1}{C_x} e^{\alpha^g r} \leq n_{\Gamma x}(r) \leq C_x e^{\alpha^g r}$$

for all $r \geq 0$, where

$$n_{\Gamma x}(r) := \#\{\gamma x \in \Gamma x \mid d(\gamma x, x_0) \leq r\}.$$

Theorem 3.1. *Let (M, g) be compact Riemannian manifold of hyperbolic type, X be its universal Riemannian covering and Γ be the group of deck transformations of X with critical exponent α^g. Then, the growth function of the volume of the geodesic balls of X is of purely exponential type with α^g as exponential factor.*

Futhermore, we have :

$$\alpha^g = h_g := \lim_{r\to\infty} \frac{\log \mathrm{vol}_n(B_g(x_0, r))}{r}.$$

Proof. By lemma 3.0.8, we have $\Lambda^g(\Gamma, x) = X^G(\infty)$. Then, the Gromov hull $E(\Lambda^g(\Gamma, x))$ of $\Lambda^g(\Gamma, x)$ is equal to X. This implies that Γ is a quasi-convex cocompact subgroup of $Iso(X)$. Let Γx be an orbit of Γ in X.

For all $r \geq 0$, let $n_{\Gamma x}$ defined by :

$$n_{\Gamma x} = \#\{\gamma x \mid d(\gamma x, x_0) \leq r\}.$$

Let consider the map K_r definied by:

$$\begin{array}{rcl} K_r : \mathbb{R}_+ & \longrightarrow & \mathbb{R}_+ \\ x & \longmapsto & \begin{cases} 1 & \text{if } 0 \leq x \leq r \\ 0 & \text{if } x > r. \end{cases} \end{array}$$

Let $\mathcal{F}$ be a fundamental domain of Γ in X. We have :

$$\begin{aligned} \mathrm{vol}_n(B_g(x_0, r)) &= \int_X K_r(d(x_0, x))d\mathrm{vol}_n(x) \\ &= \sum_{\gamma\in\Gamma} \int_{\gamma\mathcal{F}} K_r(d(x_0, x))d\mathrm{vol}_n(x) \\ &= \sum_{\gamma\in\Gamma} \int_{\mathcal{F}} K_r(d(x_0, \gamma x))d\mathrm{vol}_n(x) \\ &= \int_{\mathcal{F}} \sum_{\gamma\in\Gamma} K_r(d(x_0, \gamma x))d\mathrm{vol}_n(x) \\ &= \int_{\mathcal{F}} n_{\Gamma x}(r)d\mathrm{vol}_n(x). \end{aligned}$$

Let x_1 be a fixed point in $\mathcal{F}$ and $D = \mathrm{diam}\,\mathcal{F}$. For all $\gamma \in \Gamma$ and $x \in \mathcal{F}$, we have :

$$d(\gamma x, x_0) \leq r \Longrightarrow d(\gamma x_1, x_0) \leq r + D$$

and for $r \geq D$,

$$d(\gamma x_1, x_0) \leq r - D \Longrightarrow d(\gamma x, x_0) \leq r.$$

Then,

$$n_{\Gamma x_1}(r - D) \leq n_{\Gamma x}(r) \leq n_{\Gamma x_1}(r + D) \quad \text{for all} \quad x \in \mathcal{F} \text{ and } r \geq D.$$

By lemma 3.0.9, there is a constant $C_{x_1} \geq 1$ such that :

$$\frac{1}{C_{x_1}} e^{\alpha^g(r-D)} \leq n_{\Gamma x}(r) \leq C_{x_1} e^{\alpha^g(r+D)}$$

for all $r \geq D$ and $x \in \mathcal{F}$. Then, there exist constants $a_1 > 1$ and $r_1 := D$ such that :

$$\frac{1}{a_1} \leq \frac{\mathrm{vol}_n(B_g(x_0, r))}{e^{\alpha^g r}} \leq a_1 \quad \text{for all} \quad r \geq r_1.$$

□

Corollary 3.1.1. *Let (M, g) be a compact orientable surface of genus ≥ 2 and X be its universal Riemannian covering. Then the growth function of the volume of geodesic balls of X is of pure exponential type.*

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