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Deformation Quantization of a Harmonic Oscillator in a General Non-commutative Phase Space: Energy Spectrum in Relevant Representations

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Abstract. In this paper, we discuss deformation quantization of a harmonic oscillator in a general non-commutative phase space, with both non-commuting spatial and momentum coordinates. Different representations are considered.

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1. Introduction

In recent years, there is an increasing interest in the application of non-commutative (NC) geometry to physical problems [1] in solid-state and particle physics [2], mainly motivated by the idea of a strong connection of non-commutativity with field and string theories. Besides, the evidence coming from the latter and other approaches to the issues of quantum gravity suggests that attempts to unify gravity and quantum mechanics could ultimately lead to a non-commutative geometry of spacetime. The phase space of ordinary quantum mechanics is a well-known example of non-commuting space [3]. The momenta of a system in the presence of a magnetic field are non-commuting operators as well. Since the non-commutativity between spatial and time coordinates may lead to some problems with unitarity and causality, usually only spatial non-commutativity is considered. Besides, so far quantum theory on the NC space has been extensively studied, the main approach is based on the Weyl-Moyal correspondence which amounts to replacing the usual product by a $\star$-product in the NC space. Therefore, deformation quantization has special significance in the study of physical systems on the NC space. Moreover, the problem of quantum mechanics on NC spaces can be understood in the framework of deformation quantization [4, 5]. In the same vein, some works on harmonic oscillators (ho) in the NC space from the point of view of deformation quantization have been reported in [6, 7] and references therein.

In this paper, we consider different representations of a harmonic oscillator in a general full non-commutative phase space with both the spatial and momentum coordinates being non-commutative. Indeed, non-commutativity between momenta arises naturally as a consequence of non-commutativity between coordinates, as momenta are defined to be the partial derivatives of the action with respect to the non-commutative coordinates. This work continues the investigations stated in [6, 8] and [9] devoted to the study of a quantum exactly solvable D-dimensional NC oscillator with quasi-harmonic behavior. We intend to extend previous results presenting a similar analysis to the quantum version of the two-dimensional NC system with non-vanishing momentum components. For additional details on the motivation, see [6]. The physical model resembles the Landau problem in NC quantum mechanics extensively studied in the literature. See [10] and [11] and references therein for more details. Broadly put, it is worth noticing that the description of a system of a two-dimensional (2D) ho in a full NC phase space is equivalent to that of the same ho in a constant magnetic field in some NC space.

2. Deformation Quantization (DQ) in NC phase space

Consider a $2D$ general NC phase space. The coordinates of position and momentum, $x = (x^1, x^2)$ and $p = (p^1, p^2)$, modeling the classical system of a two-dimensional ho maps into their respective quantum operators $\hat{x}$ and $\hat{p}$ giving rise to the Hamiltonian operator

$$\hat{H} = \frac{1}{2}\Big(\hat{p}_\mu \hat{p}^\mu + \hat{x}_\mu \hat{x}^\mu\Big) \tag{1}$$

with commutation relations

$$[\hat{x}^\mu, \hat{p}^\nu] = i\hbar_{\text{eff}}\delta^{\mu\nu}, \;\; [\hat{x}^\mu, \hat{x}^\nu] = i\Theta^{\mu\nu}, \;\; [\hat{p}^\mu, \hat{p}^\nu] = i\bar{\Theta}^{\mu\nu}, \quad \mu, \nu = 1, 2 \tag{2}$$

where $\Theta^{\mu\nu}$ and $\bar{\Theta}^{\mu\nu}$ are skew-symmetric tensors carrying the dimensions of $(\text{length})^2$ and $(\text{momentum})^2$, respectively. The effective Planck constant is given by

$$\hbar_{\text{eff}} = \hbar\Big(1 + \frac{\Theta^{\mu\nu}\bar{\Theta}^{\mu\nu}}{4D\hbar^2}\Big), \tag{3}$$

where $D = 2$ is the dimension of the NC space. The operators $\hat{x}^\mu$ and $\hat{p}^\nu$ can be rewritten as

$$\hat{p}^\mu = \hat{\pi}^\mu + \frac{1}{2\hbar}\bar{\Theta}^{\mu\nu}\hat{q}_\nu, \quad \hat{x}^\mu = \hat{q}^\mu - \frac{1}{2\hbar}\Theta^{\mu\nu}\hat{\pi}_\nu \tag{4}$$

in terms of $\hat{\pi}^\mu$ and $\hat{q}^\nu$ that obey the standard Weyl-Heisenberg algebra

$$[\hat{q}^\mu, \hat{\pi}^\nu] = i\hbar\delta^{\mu\nu}; \quad [\hat{q}^\mu, \hat{q}^\nu] = 0 = [\hat{\pi}^\mu, \hat{\pi}^\nu]. \tag{5}$$

In the deformation quantization theory of a classical system in the non-commutative space, one treats (x, p) and their functions as classical quantities, but replaces the ordinary product between these functions by the following generalized $\star$-product:

$$\begin{aligned}\star &= \star_{\hbar_{\rm eff}} \star_{\Theta} \star_{\bar{\Theta}} \\ &= \exp\Big[\frac{i\hbar_{\rm eff}}{2}\Big(\overleftarrow{\partial}_{x^\mu}\overrightarrow{\partial}_{p^\mu} - \overleftarrow{\partial}_{p^\mu}\overrightarrow{\partial}_{x^\mu}\Big) + \frac{i\Theta^{\mu\nu}}{2}\overleftarrow{\partial}_{x^\mu}\overrightarrow{\partial}_{x^\nu} + \frac{i\bar{\Theta}^{\mu\nu}}{2}\overleftarrow{\partial}_{p^\mu}\overrightarrow{\partial}_{p^\nu}\Big].\end{aligned} \tag{6}$$

The variables x^μ, p^μ on the NC phase space satisfy the following commutation relations similar to (2):

$$[x^\mu, p^\nu]_\star = i\hbar_{\rm eff}\delta^{\mu\nu},\ \ [x^\mu, x^\nu]_\star = i\Theta^{\mu\nu},\ \ [p^\mu, p^\nu]_\star = i\bar{\Theta}^{\mu\nu} \quad \mu,\nu = 1,2 \tag{7}$$

with the following uncertainty relations:

$$\Delta x^1 \Delta x^2 \geq \frac{\Theta}{2} \qquad \Delta p^1 \Delta p^2 \geq \frac{\bar{\Theta}}{2} \tag{8}$$

$$\Delta x^1 \Delta p^1 \geq \frac{\hbar_{\rm eff}}{2} \qquad \Delta x^2 \Delta p^2 \geq \frac{\hbar_{\rm eff}}{2}. \tag{9}$$

The first two uncertainty relations show that measurements of positions and momenta in both directions x^1 and x^2 are not independent. Taking into account the fact that Θ and $\bar{\Theta}$ have dimensions of (length)^2 and (momentum)^2 respectively, then $\sqrt{\Theta}$ and $\sqrt{\bar{\Theta}}$ define fundamental scales of length and momentum which characterize the minimum uncertainties possible to achieve in measuring these quantities. One expects these fundamental scales to be related to the scale of the underlying field theory (possible the string scale), and thus to appear as small corrections at the low-energy level or quantum mechanics. Commonly, the time evolution function for a time-independent Hamiltonian H of a system is described by the $\star$-exponential function denoted here by $e_\star^{(.)}$:

$$e_\star^{\frac{Ht}{i\hbar_{\rm eff}}} := \sum_{n=0}^{\infty} \frac{1}{n!}\Big(\frac{t}{i\hbar_{\rm eff}}\Big)^n \overbrace{H \star H \star \cdots \star H}^{\text{n times}}, \tag{10}$$

which is the solution of the following time-dependent Schrödinger equation

$$\begin{aligned} i\hbar_{\rm eff}\frac{d}{dx} e_\star^{\frac{Ht}{i\hbar_{\rm eff}}} &= H(x,p) \star e_\star^{\frac{Ht}{i\hbar_{\rm eff}}} \\ &= H\Big(x^\mu + \frac{i\hbar_{\rm eff}}{2}\partial_{p^\mu} + \frac{i\Theta^{\mu\rho}}{2}\partial_{x^\rho}, p^\nu - \frac{i\hbar_{\rm eff}}{2}\partial_{x^\nu} + \frac{i\bar{\Theta}^{\mu\sigma}}{2}\partial_{x^\sigma}\Big) e_\star^{\frac{Ht}{i\hbar_{\rm eff}}}. \end{aligned} \tag{11}$$

There corresponds the generalized $\star$-eigenvalue time-independent Schrödinger equation:

$$H \star \mathcal{W}_n = \mathcal{W}_n \star H = \mathcal{E}_n \mathcal{W}_n \tag{12}$$

where $\mathcal{W}_n$ and $\mathcal{E}_n$ stand for the Wigner function and the corresponding energy eigenvalue of the system. The Fourier-Dirichlet expansion for the time-evolution function defined as

$$e_\star^{\frac{Ht}{i\hbar_{\rm eff}}} = \sum_{n=0}^{\infty} e^{\frac{-i\mathcal{E}_n t}{\hbar_{\rm eff}}} \mathcal{W}_n \tag{13}$$

links the Wigner function to the $\star$-exponential function.

Provided the above, the operators on a NC Hilbert space can be represented by the functions on a NC phase space, where the operator product is replaced by relevant star-product. The algebra of functions of such non-commuting coordinates can be replaced by the algebra of functions on ordinary spacetime, equipped with a NC star-product. So, considering the transformations (4) and leaving out the operator symbol ˆ, we arrive at (q, π) phase space and the commutation relations change into (5), with the star-product defined in the following way.

Definition 1. Let $C^\infty(\mathbb{R}^4)$ be the space of smooth functions $f : \mathbb{R}^4 \to \mathbb{C}$. For $f, g \in C^\infty(\mathbb{R}^4)$, the formal star product is defined by

$$f \star g = f \exp\Big[\frac{i\hbar}{2}\overleftarrow{\partial}_\mu J^{\mu\nu}\overrightarrow{\partial}_\nu\Big] g. \tag{14}$$

Here the smooth functions f and g depend on the real variables q^1, q^2, π^1 and π^2, and

$$\begin{aligned}\overleftarrow{\partial}_\mu J^{\mu\nu}\overrightarrow{\partial}_\nu &= \left(\frac{\overleftarrow{\partial}}{\partial q^1}, \frac{\overleftarrow{\partial}}{\partial \pi^1}, \frac{\overleftarrow{\partial}}{\partial q^2}, \frac{\overleftarrow{\partial}}{\partial \pi^2}\right)\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} \frac{\overrightarrow{\partial}}{\partial q^1} \\ \frac{\overrightarrow{\partial}}{\partial \pi^1} \\ \frac{\overrightarrow{\partial}}{\partial q^2} \\ \frac{\overrightarrow{\partial}}{\partial \pi^2} \end{pmatrix} \\ &= \frac{\overleftarrow{\partial}}{\partial q^1}\frac{\overrightarrow{\partial}}{\partial \pi^1} - \frac{\overleftarrow{\partial}}{\partial \pi^1}\frac{\overrightarrow{\partial}}{\partial q^1} + \frac{\overleftarrow{\partial}}{\partial q^2}\frac{\overrightarrow{\partial}}{\partial \pi^2} - \frac{\overleftarrow{\partial}}{\partial \pi^2}\frac{\overrightarrow{\partial}}{\partial q^2}.\end{aligned} \tag{15}$$

Therefore, the star product $f \star g$ represents a deformation of the classical product fg. This deformation depends on the Planck constant $\hbar$. In terms of physics, the difference $f \star g - fg$ describes quantum fluctuation depending on $\hbar$. For the present case,

$$q^\mu \star \pi^\nu - q^\mu \pi^\nu = \frac{i\hbar}{2}\delta^{\mu\nu}, \quad \pi^\nu \star q^\mu - \pi^\nu q^\mu = -\frac{i\hbar}{2}\delta^{\mu\nu}. \text{ Hence } [q^\mu, \pi^\nu]_\star = i\hbar\delta^{\mu\nu}. \tag{16}$$

Let us examine now the ho eigenvalue equation in different representations.

2.1. Harmonic oscillator eigenvalue equation in annihilation and creation operator representation

Building, in the standard manner, the creation and annihilation operators of ho system as

$$a_l = \frac{q^l + i\pi^l}{\sqrt{2}} \qquad \bar{a}_l = \frac{q^l - i\pi^l}{\sqrt{2}} \quad l = 1, 2 \tag{17}$$

and using the polar coordinates such that

$$q^l = \rho_l \cos\varphi_l, \qquad \pi^l = \rho_l \sin\varphi_l, \tag{18}$$

we solve the right and left eigenvalue equations

$$\begin{aligned} a_l \star f_{mn} &= \sqrt{m\hbar} f_{m-1,n} & \bar{a}_l \star f_{mn} &= \sqrt{(m+1)\hbar} f_{m+1,n} \\ f_{mn} \star a_l &= \sqrt{(n+1)\hbar} f_{m,n+1} & f_{mn} \star \bar{a}_l &= \sqrt{n\hbar} f_{m,n-1} \end{aligned} \tag{19}$$

to find the eigenfunctions f_{mn} as

$$f_{mn} \equiv 2(-1)^m \sqrt{\frac{m!}{n!}} e^{i(n-m)\varphi_l} \Big(\frac{2\rho_l^2}{\hbar}\Big)^{\frac{n-m}{2}} L_m^{n-m}\Big(\frac{2\rho_l^2}{\hbar}\Big) e^{-\frac{\rho_l^2}{\hbar}}, \quad m,n \in \mathbb{N} \tag{20}$$

with

$$f_{00} = 2e^{-\rho_l^2/\hbar}. \tag{21}$$

L_m^{n-m} are the generalized Laguerre polynomials defined for $n = 0, 1, 2, \dots$, $\alpha > 1$, by

$$L_n^\alpha(x) = \frac{1}{n!} e^x x^{-\alpha} \frac{d^n}{dx^n}(x^{n+\alpha} e^{-x}) = \sum_{k=0}^n \frac{\Gamma(n+\alpha+1)}{\Gamma(k+\alpha+1)} \frac{(-x)^k}{k!(n-k)!}. \tag{22}$$

Then the states defined by $b_{mn}^{(4)} = f_{m_1 n_1} f_{m_2 n_2}$, where $m = (m_1, m_2)$, $n = (n_1, n_2)$, $m_1, m_2, n_1, n_2 \in \mathbb{N}$, exactly solve the right and left eigenvalue problems of the Hamiltonian $H_0 = \sum_{l=1}^2 \bar{a}_l a_l$ as

$$H_0 \star b_{mn}^{(4)} = \hbar(|m|+1) b_{mn}^{(4)} \quad \text{and} \quad b_{mn}^{(4)} \star H_0 = \hbar(|n|+1) b_{mn}^{(4)} \tag{23}$$

where $|m| = m_1 + m_2$.

2.2. Harmonic oscillator eigenvalue equation in (q, π)-representation

Now, consider the Hamiltonian (1) and use the relation (5) to re-express it with the help of variables q and π as follows:

$$H = H_0 + H_L + H_q(\bar{\Theta}) + H_\pi(\Theta) \tag{24}$$

where

$$H_0 = \frac{1}{2}\Big((q^1)^2 + (q^2)^2 + (\pi^1)^2 + (\pi^2)^2\Big) \tag{25}$$

$$H_L = -\frac{\Theta + \bar{\Theta}}{2\hbar} \vec{q} \wedge \vec{\pi} \qquad \vec{q} \wedge \vec{\pi} = q^1 \pi_2 - q^2 \pi_1 \tag{26}$$

and

$$H_q(\bar{\Theta}) = \frac{\bar{\Theta}^2}{8\hbar^2}\Big((q^1)^2 + (q^2)^2\Big) \quad H_\pi(\Theta) = \frac{\Theta^2}{8\hbar^2}\Big((\pi^1)^2 + (\pi^2)^2\Big). \tag{27}$$

It is a matter of computation to verify that the Hamiltonians H_0 and H_L $\star$-commute. Idem for the Hamiltonians H_L and $H_I = H_q(\bar{\Theta}) + H_\pi(\Theta)$. Therefore, the Hamiltonians of family $\{H_0, H_L\}$, (respectively $\{H_L, H_I\}$) can be simultaneously measured. There follow two relevant situations.

2.2.1. Case $\Theta = -\bar{\Theta}$. The Hamiltonian H can be expressed as

$$H = \Big(1 + \frac{\Theta^2}{4\hbar^2}\Big) H_0 \tag{28}$$

and the states $b_{mn}^{(4)}$ solve the right and left eigenvalue problems of H as

$$H \star b_{mn}^{(4)} = \mathcal{E}_{m0}^R b_{mn}^{(4)} \qquad \mathcal{E}_{m0}^R = \hbar\big(1 + \big(\Theta^2/4\hbar^2\big)\big)(|m|+1) \tag{29}$$

and

$$b_{mn}^{(4)} \star H = \mathcal{E}_{0n}^L b_{mn}^{(4)} \qquad \mathcal{E}_{0n}^L = \hbar\big(1 + \big(\Theta^2/4\hbar^2\big)\big)(|n|+1) \tag{30}$$

where $m = (m_1, m_2)$, $n = (n_1, n_2)$, $m_1, m_2, n_1, n_2 \in \mathbb{N}$, $|m| = m_1 + m_2$.

2.2.2. Case $\Theta = \bar{\Theta}$. The Hamiltonian H can be rewritten as

$$H = \Big(1 + \frac{\Theta^2}{4\hbar^2}\Big)H_0 - \frac{\Theta}{\hbar}\overrightarrow{q} \wedge \overrightarrow{\pi}. \quad (31)$$

The eigenvectors of H_0 and H_L are eigenvectors of H, (as they commute each with other), with eigenvalues

$$\mathcal{E}^R_{mn} = \hbar\Big(1 + \frac{\Theta^2}{4\hbar^2}\Big)(|m| + 1) - (|n| - |m|)\Theta \quad (32)$$

and

$$\mathcal{E}^L_{mn} = \hbar\Big(1 + \frac{\Theta^2}{4\hbar^2}\Big)(|n| + 1) - (|m| - |n|)\Theta \quad (33)$$

corresponding to the right and left eigenvalue equations

$$H \star b^{(4)}_{mn} = \mathcal{E}^R_{mn} b^{(4)}_{mn} \quad (34)$$

and

$$b^{(4)}_{mn} \star H = \mathcal{E}^L_{mn} b^{(4)}_{mn}. \quad (35)$$

2.3. Harmonic oscillator eigenvalue equation in a general (q, π)-representation

The problem to be solved is equivalent to that of a two-dimensional Landau problem in a symmetric gauge on a non-commutative space. Indeed, the Hamiltonian H can be re-transcribed as

$$H = \frac{\alpha^2}{2}\Big((q^1)^2 + (q^2)^2\Big) + \frac{\beta^2}{2}\Big((\pi^1)^2 + (\pi^2)^2\Big) - \gamma\overrightarrow{q} \wedge \overrightarrow{\pi} =: H_0^\natural + H_L \quad (36)$$

where

$$\alpha^2 = 1 + \frac{\bar{\Theta}^2}{4\hbar^2}, \qquad \beta^2 = 1 + \frac{\Theta^2}{4\hbar^2}, \qquad \gamma = \frac{\Theta + \bar{\Theta}}{2\hbar} \quad (37)$$

Remark that the Hamiltonian terms $H_0^\natural$ and H_L commute. Therefore, the eigenvectors of $\{H_0^\natural, H_L\}$ are automatically eigenvectors of H. As matter of convenience, to solve the Schrödinger eigen-equation, let us choose the polar coordinates

$$q^1 = \rho\cos\varphi \qquad q^2 = \rho\sin\varphi \quad (38)$$

and assume the variable separability to write

$$\tilde{f}(\rho, \varphi) = \xi(\rho)e^{ik\varphi}, \quad k = 0, \pm 1, \pm 2, \cdots \quad (39)$$

Then, from the static Schrödinger equation on NC space, $H \star \tilde{f}(\rho, \varphi) = \mathcal{E}\tilde{f}(\rho, \varphi)$, we deduce the radial equation as follows:

$$\Big[- \frac{\hbar^2\beta^2}{2}\Big(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho}\Big) + \frac{\alpha^2}{2}\rho^2 - \gamma\hbar k\Big]\xi(\rho, \varphi) = \mathcal{E}\xi(\rho, \varphi) \quad (40)$$

yielding the spectrum of H under the form

$$\mathcal{E} = \hbar\frac{\alpha^2}{\beta^2}(n + 1) - \hbar\gamma k, \quad n = 0, 1, 2, \dots \quad (41)$$

with

$$\xi(\rho, \varphi) \propto e^{-\frac{\alpha}{\hbar\beta}\rho^2} H_n\left(\frac{\alpha}{\hbar\beta}\rho^2\right). \quad (42)$$

The last term of the energy spectrum $\mathcal{E}$ falls down when $\gamma = 0$, i.e., $\Theta = -\bar{\Theta}$. In this case, $\alpha^2 = \beta^2$ and we recover the discrete spectrum of the usual two-dimensional harmonic oscillator as expected. The results obtained here can be reduced to specific expressions reported in the literature [6] for particular cases. Besides, the formalism displayed in this work permits to avoid the appearance of infinite degeneracy of states observed when $\hbar_{\text{eff}}^2 - \Theta\bar{\Theta} = 0$ in [10] where the phase space is divided into two phases based on the following conditions on the deformation parameters:

- Phase I for $\hbar_{\text{eff}}^2 - \Theta\bar{\Theta} > 0$
- Phase II for $\hbar_{\text{eff}}^2 - \Theta\bar{\Theta} < 0$.

Finally, let us mention that the direct computation of the energy spectrum from the relation (24) instead of (36) introduces an unexpectable feature, i.e., the energy spectrum depends on the phase space variables as it should not be with respect to the study performed in [11]. Such a pathology is generated by the phase space variable dependence of the commutator

$$[H_0, H_I]_\star = i\frac{\Theta^2 - \bar{\Theta}^2}{4\hbar}(q^1\pi^1 + q^2\pi^2). \tag{43}$$

This could explain why previous investigations (see [6], [12] and [13] and references therein) were restricted to the cases $\Theta = \pm\bar{\Theta}$.

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