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On abstract indefinite concave–convex problems and applications to quasilinear elliptic equations

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Abstract. In this work we study the existence of critical points of an **abstract** C^1 functional J defined in a reflexive Banach space X. This functional is of the form

$$J(u) = \frac{1}{p}E(u) - \frac{1}{r}A(u) - \frac{1}{q}B(u),$$

with E, A, B positive-homogeneous **indefinite** functional of degree p, q, r respectively and $1 < p < q < r$. The critical points are found by minimization along several subsets of the Nehari manifold associated to J. We apply these results to various quasilinear elliptic problems, as for instance, the following p-laplacian concave–convex problem with Steklov boundary conditions on a bounded regular domain

$$\begin{cases} -\Delta_p u + V(x)u^{p-1} = 0 & \text{in } \Omega; \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = \lambda a(x)u^{r-1} + b(x)u^{q-1} & \text{on } \partial\Omega; \\ u > 0 \text{ in } \Omega, \end{cases}$$

with given functions a, b, V possibly indefinite and $1 < r < p < q$. We also apply our abstract result for a concave–convex quasilinear problem associated to the p-bilaplacian.

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Keywords. Concave–convex problem, Nehari manifold, Dirichlet–Steklov boundary condition, Elliptic problem, Non-coerciveness, p-Laplacian, p-bilaplacian, Indefinite weights.

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1. Introduction

After the celebrated paper of Ambrosetti-Brézis-Cerami [1] on the solvability of the elliptic problem

$$\begin{cases} -\Delta u = f_\lambda(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega, \end{cases}$$

for $f_\lambda(u) = \lambda u^{r-1} + u^{q-1}$ and $1 < r < 2 < q$, there have been a huge amount of research on this type of equations and the effects on multiplicity of the concave $(1 < r < 2)$-convex $(2 < q)$ nonlinearity. In [3–5] the authors studied the previous equation with a concave–convex forcing term $f_\lambda(u) = \lambda a(x)u^{r-1} + b(x)u^{q-1}$ with changing-sign weights $a(x), b(x)$ by using the so called *Nehari manifold and the fibering map* associated to the problem. This approach has proved to be very useful to deal with this type of problem and have become a subject of research on its own. The results of [1] were partially generalized to the $p-$laplacian operator under Dirichlet boundary conditions in [2,8,11,12,21]. Soon later, other boundary conditions have been considered, as for instance in [17]. Simultaneously some attention has been accorded to quasilinear problems that *are non coercive*. The first work in this direction was the study of the spectrum of the operator $-\Delta_p u + V(x)|u|^{p-2}u$ with Dirichlet boundary conditions and V an indefinite bounded weight, see [7]. In [16] the author studied the concave–convex problem

$$\begin{cases} -\Delta_p u + V(x)u^{p-1} = \lambda a(x)u^{r-1} + b(x)u^{q-1} & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega; \\ u > 0 & \text{in } \Omega, \end{cases}$$

by minimization on the Nehari set and obtained, under various "coerciveness" conditions related to V, a and b, the existence of up to *four solutions*: two solutions satisfying the condition $E_V(u) > 0$ with

$$E_V(u) := \int_\Omega (|\nabla u|^p + V(x)|u|^p)\, dx,$$

and two more solutions satisfying $E_V(u) < 0$.

Our goal in this work is to generalize the results of [16] for the same quasilinear operator, say $-\Delta_p u + V(x)|u|^{p-2}u$, but with different boundary conditions. Precisely, let Ω be a bounded smooth domain of class $C^{2,\alpha}$ $(0 < \alpha < 1)$ with outward unit normal ν on the boundary $\partial\Omega$ and $\Delta_p u := div\,(|\nabla u|^{p-2}\nabla u)$ is the well known $p-$laplacian operator. The functions $V \in L^\infty(\Omega)$ and $(a, b) \in (C^s(\partial\Omega))^2$, for some $s \in (0, 1)$, are allowed to change sign. The real number λ is a positive parameter. We will ask the exponents r, q to satisfy $1 < r < p < q < p_*$ where $p_* = \frac{p(N-1)}{(N-p)^+}$ is the critical exponent for the trace operator $W^{1,p}(\Omega) \to L^s(\partial\Omega, d\rho)$, and ρ denotes the restriction to $\partial\Omega$ of the $(N-1)$-Hausdorff measure, which coincides with the usual Lebesgue surface measure as $\partial\Omega$ is regular enough. We will consider the following quasilinear elliptic

Problem I:

$$\begin{cases} -\Delta_p u + V(x)u^{p-1} = 0 & \text{in } \Omega; \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = \lambda a(x)u^{r-1} + b(x)u^{q-1} & \text{on } \partial\Omega; \\ u > 0 \text{ in } \overline{\Omega}, \end{cases} \tag{1.1}$$

and **Problem II**:

$$\begin{cases} -\Delta_p u + V(x)u^{p-1} = \lambda a(x)u^{r-1} & \text{in } \Omega; \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = b(x)u^{q-1} & \text{on } \partial\Omega; \\ u > 0 \text{ in } \overline{\Omega}. \end{cases} \tag{1.2}$$

The search of solutions for these quasilinear problems can be stated in an **abstract form** as the search of critical points of a functional J defined on a Banach space X, which will take the form

$$J(u) = \frac{1}{p}E(u) - \frac{1}{r}A(u) - \frac{1}{q}B(u), \tag{1.3}$$

where E, A, B are possibly indefinite but positive-homogeneous functional of degree p, r, q respectively, with $1 < r < p < q$. We will prove several existence and multiplicity results on critical points of J by minimizing J along several subsets of the **Nehari** manifold

$$\mathcal{N} = \{u \in X \setminus \{0\};\ \langle J'(u), u\rangle = 0\}$$

associated to J. These existence results for a general functional E, A, B and X can be applied in many cases, as for instance for the *p-bilaplacian operator* with Navier boundary conditions, that is the following **Problem III**:

$$\begin{cases} \Delta_p^2 u - c|u|^{p-2}u = \lambda a(x)|u|^{r-2}u + b(x)u^{q-2}u & \text{in } \Omega; \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

with $c \in \mathbb{R}$. The p-bilaplacian operator is defined as $\Delta_p^2 u := \Delta(|\Delta u|^{p-2}\Delta u)$ and it has received recently some attention. In the case $p = 2$ the authors in [22] generalize the Ambrosetti-Brézis-Cerami problem in the case $c = 0$ (coercive case), indefinite weight a and non-negative weight b. See also [15] for similar results in $\mathbb{R}^N$ or [13] for Dirichlet boundary conditions ($u = \nabla u = 0$) and $p \neq 2$.

This paper is organized as follows. In Sect. 2 we describe the Nehari set, the fibering map and the different "sign-subsets" of the Nehari set that will be used to find critical points of the C^1 functional J which is defined in (1.3). In Sects. 3 and 4 we prove four different critical points theorems of the functional J in the Nehari set, c.f. Theorems 3.3, 3.4, 4.1 and 4.2. The hypothesis needed to apply these general theorems concern the coerciveness of E along the sign-subsets of the Nehari set described in Sect. 2. In Sect. 5 we present various conditions in terms of eigenvalues-like numbers that will imply the required coerciveness and then we prove some existence results (see Theorems 5.1 and 5.7). In Sect. 6 we apply the theorem of the previous section to state some existence and multiplicity results for problems I, II and III.

2. The Nehari set for a concave–convex functional

Let $(X, \|\cdot\|)$ be a *reflexive* Banach space and $E, A, B \in C^1(X, \mathbb{R})$. Let us assume that for some $1 < r < p < q$ it holds

$$E(tu) = t^p E(u), \quad A(tu) = t^r A(u), \quad B(tu) = t^q B(u), \quad \forall (t,u) \in \mathbb{R}^+ \times X. \tag{2.1}$$

The following hypothesis will also be assumed:

(H1) $\forall (u_n)_{n\in\mathbb{N}}, u_n \in X$, if $u_n \rightharpoonup u$ for some $u \in X$ then there exists a subsequence u_{n_k} such that

$$A(u_{n_k}) \to A(u) \text{ and } B(u_{n_k}) \to B(u).$$

(H2) E is bounded on bounded sets (i.e. $\sup\limits_{u\in X, \|u\|=1} |E(u)| < \infty$) and it is weakly lower semi-continuous.

(H3) $\forall (u_n)_{n\in\mathbb{N}}, u_n \in X$, if $u_n \rightharpoonup u$ for some $u \in X$ and $E(u_n) \to E(u)$ then $u_n \to u$.

By "$\rightharpoonup$" we denote the weak convergence in X. Let us consider the functional J defined as

$$J(u) = \frac{1}{p}E(u) - \frac{1}{r}A(u) - \frac{1}{q}B(u)$$

and look for solutions of the problem $J'(u) = 0$. Since J may be unbounded from below on the set $\{u \in X \ ; \ B(u) > 0\}$, it is useful to consider the functional J restricted to the so called **Nehari set**

$$\mathcal{N} := \{u \in X \setminus \{0\} \ ; \ N(u) := \langle J'(u), u\rangle = 0\},$$

where $\langle \cdot, \cdot \rangle$ is the usual duality map defined on $X' \times X$. Thus $u \in \mathcal{N}$ if and only if $u \neq 0$ and

$$E(u) = A(u) + B(u).$$

Let us introduce the **fibering maps** associated to J. For $u \in X \setminus \{0\}$, we define the function

$$\begin{aligned} J_u : \ (0,\infty) &\longrightarrow \quad \mathbb{R} \\ t \quad &\longmapsto J_u(t) := J(tu). \end{aligned}$$

Consequently $u \in \mathcal{N}$ if and only if $J_u'(1) = 0$ or yet, $tu \in \mathcal{N}$ (with $t > 0$) if and only if $J_u'(t) = 0$. The notation J_u', J_u'' stands here for $\frac{dJ_u}{dt}, \frac{d^2 J_u}{dt^2}$ resp. Furthermore, since $J_u'(u) = 0$ for $u \in \mathcal{N}$ we can write

$$\begin{aligned} J_u''(1) = &\ (p-q)E(u) - (r-q)A(u) \\ = &\ (p-r)E(u) - (q-r)B(u). \end{aligned} \tag{2.2}$$

It is standard to split $\mathcal{N}$ into three sets that, roughly speaking, correspond to local minima, local maxima and inflexion points of J_u:

$$\mathcal{N}^+ := \left\{u \in X\ ;\ J_u^{'}(1) = 0, J_u^{''}(1) > 0\right\},$$
$$\mathcal{N}^- := \left\{u \in X\ ;\ J_u^{'}(1) = 0, J_u^{''}(1) < 0\right\},$$
$$\mathcal{N}_0 := \left\{u \in X\ ;\ J_u^{'}(1) = 0, J_u^{''}(1) = 0\right\}.$$

We also introduce the following "sign-subsets":

$$\mathcal{A}^{\pm} := \{u \in X; A(u) \gtrless 0\}, \quad \mathcal{A}_0 = \{u \in X; A(u) = 0\},$$
$$\mathcal{B}^{\pm} := \{u \in X; B(u) \gtrless 0\}, \quad \mathcal{B}_0 = \{u \in X; B(u) = 0\},$$
$$\mathcal{E}^{\pm} := \{u \in X; E(u) \gtrless 0\}, \quad \mathcal{E}_0 = \{u \in X; E(u) = 0\}.$$

Finally we denote $\mathcal{A}_0^{\pm} = \mathcal{A}^{\pm} \cup \mathcal{A}_0$ and similarly for $\mathcal{B}_0^{\pm}$ and $\mathcal{E}_0^{\pm}$. We stress here that, without further assumptions on E, A, B, the previous sets can be void.

Next we state the following property of $\mathcal{N}$ that follows straight from hypothesis (**H1**)–(**H3**).

Lemma 2.1. *Let u_n be a sequence in $\mathcal{N}$ such that $u_n \rightharpoonup 0$. Then $u_n \to 0$.*

Let us give a result on the boundedness of $\mathcal{N}$. We recall that a subset C of X is called a *cone* if $tx \in C, \forall (t, x) \in \mathbb{R}^+ \times C$.

Lemma 2.2. *Let $\mathcal{C} \subset X$ be weakly closed cone such that*

$$(C \cap \mathcal{B}_0) \setminus \{0\} \subset \mathcal{E}^+. \tag{2.3}$$

Then

(i) the set $\mathcal{N}^+ \cap \mathcal{C}$ is bounded;

(ii) if $\sup_{\mathcal{C}\cap\mathcal{N}} J(u) < \infty$ then $\mathcal{N} \cap \mathcal{C}$ is bounded.

Proof. (i) Assume by contradiction that there is an unbounded sequence $u_n \in \mathcal{N}^+ \cap \mathcal{C}$. Take $v_n = \frac{u_n}{\|u_n\|} \in C$ and $v_0 \in C$ such that $v_n \rightharpoonup v_0$. From the fact that $u_n \in \mathcal{N}$ we have

$$\frac{E(v_n)}{\|u_n\|^{q-p}} = \frac{A(v_n)}{\|u_n\|^{q-r}} + B(v_n)$$

and passing to the limit we conclude that $B(v_0) = 0$. We have used here that E is bounded on bounded sets [i.e. (**H2**)]. From the fact that $u_n \in \mathcal{N}^+$ we have

$$E(v_n) < \frac{q-r}{q-p} \frac{A(v_n)}{\|u_n\|^{p-r}}$$

and passing to the limit it comes that $E(v_0) \le 0$. The possibility of $v_0 = 0$ is ruled out by the fact that, in that case, $0 = E(v_0) = \liminf_{n\to\infty} E(v_n)$ and therefore, by (**H3**), we will have $v_n \to v_0 = 0$. This is a contradiction with the property $\|v_n\| = 1$. Thus $v_0 \ne 0$ and we get a contradiction with the hypothesis (2.3) of the lemma.

(ii) Assume that $u_n \in \mathcal{N} \cap \mathcal{C}$ is a sequence satisfying $\|u_n\| \to +\infty$ and denote $v_n = \frac{u_n}{\|u_n\|}$. Since the sequence v_n is bounded, there exists $v_0 \in X$ and a subsequence v_{n_k} such that $v_{n_k} \rightharpoonup v_0$. From the fact that $u_{n_k} \in \mathcal{N}$ we have, as previously, $0 = B(v_0)$. On the other hand

$$\frac{J(u_{n_k})}{\|u_{n_k}\|^p} = \left(\frac{1}{p} - \frac{1}{q}\right) E(v_{n_k}) - \left(\frac{1}{r} - \frac{1}{q}\right) \frac{A(v_{n_k})}{\|u_{n_k}\|^{p-r}}$$

therefore passing to the limit we get

$$E(v_0) \leq \liminf_{k \to +\infty} E(v_{n_k}) \leq 0$$

because $J(u_n)$ is uniformly bounded from above. If $v_0 = 0$ hence $E(v_0) = 0 = \liminf_{k \to +\infty} E(v_{n_k})$ and, from hypothesis (**H3**), $v_{n_k} \to v_0 = 0$. This is impossible because $\|v_{n_k}\| = 1$ for all k. Thus $v_0 \neq 0$ and we get again a contradiction with the hypothesis (2.3) of the lemma. □

We also have the following property:

Lemma 2.3. *Let $\mathcal{C} \subset X$ be weakly closed cone and assume that*

$$C \cap \mathcal{A}_0 \setminus \{0\} \subset \mathcal{E}^+. \tag{2.4}$$

Then $\mathcal{N}^- \cap \mathcal{C}$ has no sequence $u_n \rightharpoonup 0$.

Proof. Assume by contradiction that there is a sequence $u_n \in \mathcal{N}^- \cap \mathcal{C}$ such that $u_n \rightharpoonup 0$ in X. From the fact that $u_n \in \mathcal{N}$ we have $0 = E(0) \leq \liminf E(u_n) = \liminf A(u_n) + B(u_n)$ so $u_n \to 0$ on X. Take $z_n := \frac{u_n}{\|u_n\|} \in C$ and assume that for some $z_0 \in X$ we have $z_n \rightharpoonup z_0$, $A(z_n) \to A(z_0)$ and $B(z_n) \to B(z_0)$. By using that $u_n \in \mathcal{N}$ we have

$$A(z_n) = E(z_n)\|u_n\|^{p-r} - B(z_n)\|u_n\|^{q-r}$$

and passing to the limit it comes $A(z_0) = 0$. Besides by using that $u_n \in \mathcal{N}^-$ and (2.2) we have

$$E(z_n) \leq \frac{q-r}{p-r} B(z_n)\|u_n\|^{q-p},$$

and passing to the limit

$$E(z_0) \leq \liminf E(z_n) \leq 0.$$

Notice that the possibility $z_0 = 0$ is excluded because, in that case, we would have $0 = E(z_0) = \liminf E(z_n)$ which will imply that $z_n \to z_0 = 0$, a contradiction with the fact that $\|z_n\| = 1$. Thus we have proved that $z_0 \in C \cap \mathcal{A}_0 \setminus \{0\}$. Then from the hypothesis (2.4) of this lemma it comes that $E(z_0) > 0$, a contradiction. □

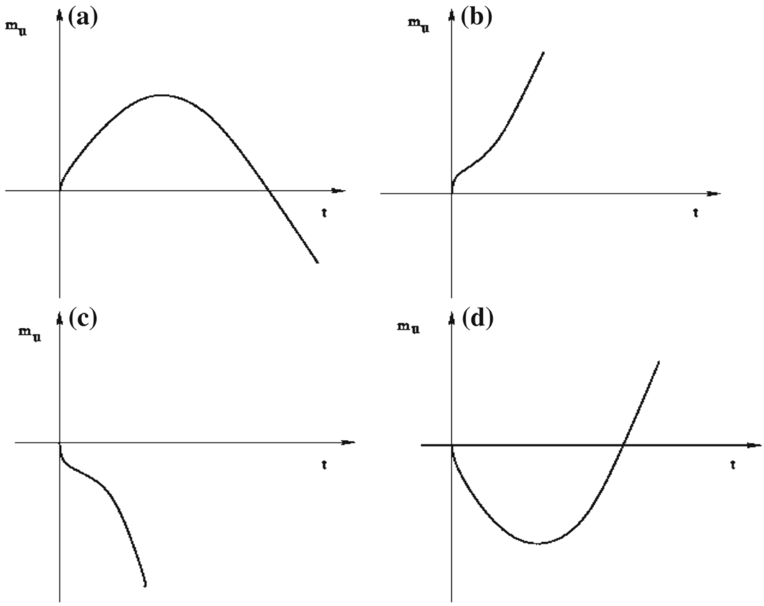

Figure 1. Possible forms of m_u

2.1. The fibering map

Let us give a complete description of the behaviour of J_u according to the sign of $A(u)$, $B(u)$ and $E(u)$. Let us write

$$J_u'(t) = t^{p-1}E(u) - t^{r-1}A(u) - t^{q-1}B(u) = t^{r-1}[m_u(t) - A(u)],$$

where

$$m_u(t) := t^{p-r}E(u) - t^{q-r}B(u). \tag{2.5}$$

Clearly, for $t > 0$, $tu \in \mathcal{N}$ if and only if t is a solution of the equation

$$m_u(t) = A(u). \tag{2.6}$$

If for $u \in X\backslash\{0\}$ and $t > 0$ one has that $J_u'(t) = 0$ then $J_u''(t) = t^{r-1}m_u'(t)$. Consequently $tu \in \mathcal{N}^+$ if and only if $m_u'(t) > 0$ (similar results for $\mathcal{N}^-$ and $\mathcal{N}_0$). In order to study the resolvability of (2.6) let us describe the variation of the function $t \mapsto m_u(t)$ for any $u \notin \mathcal{E}_0 \cap \mathcal{B}_0$. Four possible pictures of the graph of m_u can be drawn:

Case I : $E(u) > 0$ and $B(u) > 0$ in Fig. 1a;
Case II : $E(u) \geq 0$ and $B(u) \leq 0$ in Fig. 1b;
Case III : $E(u) \leq 0$ and $B(u) \geq 0$ in Fig. 1c;
Case IV : $E(u) < 0$ and $B(u) < 0$ in Fig. 1d.

We shall now describe the nature of the fibering maps for all possible signs of $A(u)$, $B(u)$ and $E(u)$. The following behaviour of the function J_u follows from the previous description of the function m_u. As above, we will assume that $u \notin \mathcal{E}_0 \cap \mathcal{B}_0$.

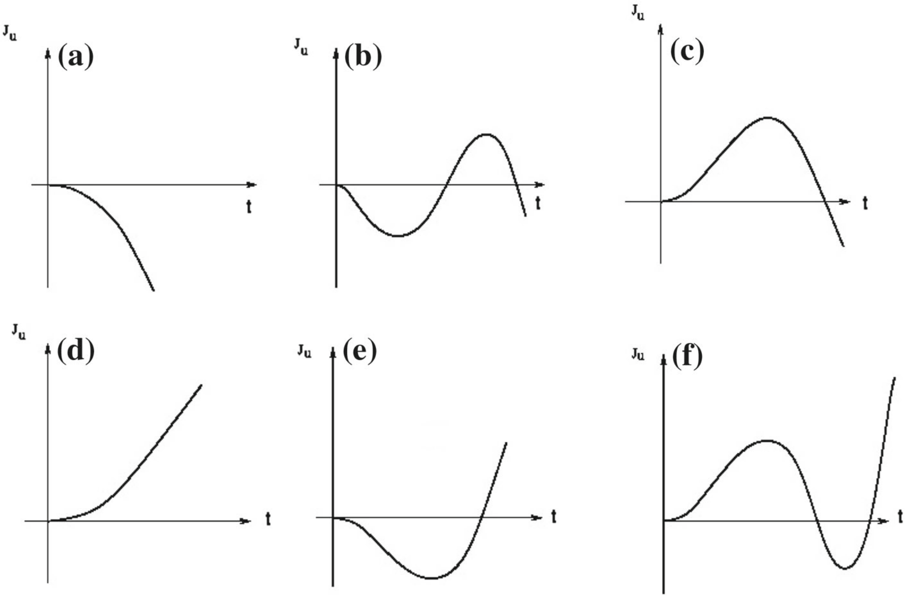

FIGURE 2. Possible forms of J_u

Case 1 : $E(u) > 0$ and $B(u) > 0$.

In this case the function m_u has graph as shown in Fig. 1a.

Case 1.1 : If $A(u) > 0$ and $A(u) < \max_{t>0} m_u(t)$ then it is clear that there are exactly two solutions $0 < t_1(u) < t_2(u)$ of (2.6) with $m'_u(t_2(u)) < 0 < m'_u(t_1(u))$. Thus there are exactly two multiples of u lying in $\mathcal{N}$, namely $t_1(u)u \in \mathcal{N}^+$ and $t_2(u)u \in \mathcal{N}^-$. It follows that J_u has exactly two critical points, a local minimum at $t_1(u)$ and a local maximum at $t_2(u)$. Moreover J_u is decreasing in $(0, t_1(u))$ and in $(t_2(u), \infty)$, increasing in (t_1, t_2), see Fig. 2b.

Case 1.2 : If $A(u) \leq 0$. Then using the graph of m_u in Fig. 1a, we deduce that there exists one positive solution of (2.6). Consequently J_u has graph as shown in Fig. 2c and there is a unique value $t(u) > 0$ such that $t(u)u \in \mathcal{N}$. Moreover $m'_u(t(u)) < 0$, so $t(u)u \in \mathcal{N}^-$ and the fibering maps J_u has a unique critical point which is a local maximum.

Case 2 : $E(u) \geq 0$ and $B(u) \leq 0$.

In this case the function m_u is an increasing function of t (see Fig. 1b).

Case 2.1 : If $A(u) > 0$, then m_u has a graph as in Fig. 1b and J_u has a graph as shown in Fig. 2e. It is clear that there is exactly one solution of (2.6), i.e. there is a unique $t(u) > 0$ such that $t(u)u \in \mathcal{N}$. Moreover $m'_u(t(u)) > 0$ (since m_u is an increasing function of t) and so $t(u)u \in$

$\mathcal{N}^+$. Thus the fibering map J_u has a unique critical point which is a local minimum, as shown in Fig. 2e.

Case 2.2 : If $A(u) \leq 0$ then the function J_u is increasing functions of t and so has graph as shown in Fig. 2d. Consequently (2.6) has no solution, for all t and thus no multiple of u lies in $\mathcal{N}$.

Case 3 : $E(u) \leq 0$ and $B(u) \geq 0$.

In this case the function m_u is a decreasing function of t and has graph as shown in Fig. 1c.

Case 3.1 : If $A(u) < 0$ then (2.6) has a unique solution. Since J_u must have graph as shown in Fig. 2c, we conclude again that there is a unique $t(u) > 0$ such that $t(u)u \in \mathcal{N}$ and since $m'_u(t(u)) < 0$ in this case, we deduce that $t(u)u \in \mathcal{N}^-$. Hence the fibering map J_u has a unique critical point which is a local maximum.

Case 3.2 : If $A(u) \geq 0$ then (2.6) has no solution. Moreover J_u is a decreasing function of t and has graph as shown in Fig. 2a. Thus in this case no multiple of u lies in $\mathcal{N}$.

Case 4 : $E(u) < 0$ and $B(u) < 0$.

In this case m_u has graph as shown in Fig. 1d.

Case 4.1 : If $A(u) < 0$ and $A(u) > \min\limits_{t>0} m_u(t)$ then J_u has graph as shown in Fig. 2f. In this case there are exactly two solutions $t_1(u) < t_2(u)$ of (2.6) with $m'_u(t_1(u)) < 0 < m'_u(t_2(u))$. Thus there are exactly two multiples of u which belong to $\mathcal{N}$, namely $t_1(u) \in \mathcal{N}^-$ and $t_2(u) \in \mathcal{N}^+$. It follows that J_u has exactly two critical points, a local maximum at $t = t_1(u)$ and a local minimum at $t = t_2(u)$. Furthermore J_u is increasing in $(0, t_1)$, decreasing in (t_1, t_2) and increasing in (t_2, ∞), as in Fig. 2f.

Case 4.2 : If $A(u) \geq 0$ then (2.6) has a unique solution and J_u has graph as shown in Fig. 2e. Thus there is a unique value $t(u) > 0$ such that $t(u)u \in \mathcal{N}$. Since $m'_u(t(u)) > 0$, we deduce that $t(u)u \in \mathcal{N}^+$ and consequently J_u has a unique critical point which is a local minimum as shown in Fig. 2e.

It comes out from this calculation that

Proposition 2.4. (i) *If either* $\mathcal{A}^+ \cap \mathcal{B}_0^- \cap \mathcal{E}^+ \neq \emptyset$ *or*

$$\Lambda^+ := \{u \in \mathcal{A}^+ \cap \mathcal{B}^+ \cap \mathcal{E}^+ \ ; \ A(u) < \max_{t>0} m_u(t)\} \neq \emptyset$$

then $\mathcal{N}^+ \cap \mathcal{E}^+ \neq \emptyset$.

(ii) *If either* $\mathcal{A}_0^- \cap \mathcal{B}^+ \cap \mathcal{E}^+ \neq \emptyset$ *or* $\Lambda^+ \neq \emptyset$ *then* $\mathcal{N}^- \cap \mathcal{E}^+ \neq \emptyset$.

(iii) *If* $\Lambda^- := \{u \in \mathcal{A}^- \cap \mathcal{B}^- \cap \mathcal{E}^- \ ; \ A(u) > \min\limits_{t>0} m_u(t)\} \neq \emptyset$ *then* $\mathcal{N}^+ \cap \mathcal{A}^- \neq \emptyset$.

(iv) *If* $\Lambda^- \neq \emptyset$ *then* $\mathcal{N}^- \cap \mathcal{B}^- \neq \emptyset$.

A simple calculation shows that the maximum of m_u in case I is

$$\max_{t>0} m_u(t) = \left(\frac{q-p}{q-r}\right)\left(\frac{p-r}{q-r}\right)^{\frac{p-r}{q-p}} \frac{E(u)^{\frac{q-r}{q-p}}}{B(u)^{\frac{p-r}{q-p}}}$$

and the minimum of m_u in case IV is

$$\min_{t>0} m_u(t) = -\left(\frac{q-p}{q-r}\right)\left(\frac{p-r}{q-r}\right)^{\frac{p-r}{q-p}} \frac{(-E(u))^{\frac{q-r}{q-p}}}{(-B(u))^{\frac{p-r}{q-p}}}\ .$$

The maximum (resp. the minimum) of m_u is achieved at the point

$$t_*(u) := \left(\frac{(p-r)E(u)}{(q-r)B(u)}\right)^{\frac{1}{q-p}}. \tag{2.7}$$

Let us denote

$$\lambda_*^+ := \inf_{u\in\mathcal{A}^+\cap\mathcal{B}^+\cap\mathcal{E}^+} \frac{\max_{t>0} m_u(t)}{A(u)} \geq 0 \tag{2.8}$$

in case I and

$$\lambda_*^- := \inf_{u\in\mathcal{A}^-\cap\mathcal{B}^-\cap\mathcal{E}^-} \frac{\min_{t>0} m_u(t)}{A(u)} \geq 0 \tag{2.9}$$

in case IV. From the previous discussion it follows trivially that:

Lemma 2.5. *(i) If $\lambda_*^+ > 1$ then $\mathcal{N}_0 \cap \mathcal{E}^+ = \emptyset$. (ii) If $\lambda_*^- > 1$ then $\mathcal{N}_0 \cap \mathcal{E}^- = \emptyset$.*

3. Local minimizers of J restricted to the Nehari set and to $\mathcal{E}^+$

Our purpose in this section is to prove that, under some suitable assumptions on $\mathcal{A}^\pm, \mathcal{B}^\pm$ and $\mathcal{E}^\pm$, the functional J is bounded below and achieves its infimum on some of the sign subsets of $\mathcal{N}$ described in cases I to IV of Sect. 2. This will provide us critical points for J, as a consequence of the following well known result:

Lemma 3.1. *Suppose that u is a local minimiser of J restricted to $\mathcal{N}$. If $u \notin \mathcal{N}^0$ then u is a critical point of J relative to X.*

Proof. Since u is a minimiser of J on $\mathcal{N}$, there exists $\gamma \in \mathbb{R}$ (Lagrange multiplier) such that

$$J'(u) = \gamma N'(u). \tag{3.1}$$

Thus in particular we have

$$\langle J'(u), u\rangle = \gamma\langle N'(u), u\rangle,$$

which implies that $\gamma\langle N'(u), u\rangle = 0$ because $0 = N(u) = \langle J'(u), u\rangle$ (since $u \in \mathcal{N}$). Moreover

$$\begin{aligned}\langle N'(u), u\rangle &= pE(u) - rA(u) - qB(u)\\ &= (p-r)E(u) - (q-r)B(u)\\ &= J_u''(1).\end{aligned}$$

Consequently, if $u \notin \mathcal{N}^0$, that is $J_u''(1) \neq 0$,then $\gamma = 0$ and we conclude from (3.1) that u is a critical point of J. ∎

Let us rewrite the functional J for $u \in \mathcal{N}$ in two different forms:

$$\begin{aligned} J(u) &= \left(\frac{1}{p} - \frac{1}{r}\right) E(u) + \left(\frac{1}{r} - \frac{1}{q}\right) B(u) \\ &= \left(\frac{1}{p} - \frac{1}{q}\right) E(u) - \left(\frac{1}{r} - \frac{1}{q}\right) A(u). \end{aligned} \tag{3.2}$$

We then observe the following

Lemma 3.2. (a) $J(u) > 0$ *for all* $u \in (\mathcal{N}^- \cap \mathcal{A}_0^-) \cup (\mathcal{N}^- \cap \mathcal{B}_0^+) \cup (\mathcal{N}^- \cap \mathcal{E}_0^-)$;
(b) $J(u) < 0$ *for all* $u \in (\mathcal{N}^+ \cap \mathcal{A}_0^+) \cup (\mathcal{N}^+ \cap \mathcal{B}_0^-) \cup (\mathcal{N}^+ \cap \mathcal{E}_0^+)$.

Proof. From (2.2) we deduce the following inequalities

(a) If $u \in \mathcal{N}^-$ then

$$\frac{q-r}{q-p} A(u) < E(u) < \frac{q-r}{p-r} B(u). \tag{3.3}$$

Hence

$$J(u) > \max\left\{\frac{(q-r)(r-p)}{prq} A(u), \frac{(q-p)(r-p)}{pqr} E(u), \frac{(q-p)(q-r)}{pqr} B(u)\right\},$$

(b) Similarly, if $u \in \mathcal{N}^+$ then

$$\frac{q-r}{p-r} B(u) < E(u) < \frac{q-r}{q-p} A(u). \tag{3.4}$$

Hence

$$J(u) < \min\left\{\frac{(q-r)(r-p)}{pqr} A(u), \frac{(q-p)(r-p)}{pqr} E(u), \frac{(q-p)(q-r)}{pqr} B(u)\right\}.$$

□

3.1. Minimizing J along $\mathcal{N}^+$

A first critical point of J can be found on $\mathcal{N}^+ \cap \mathcal{E}^+$, provided this set is not empty and some "coerciveness" conditions:

Theorem 3.3. *Assume that* $\mathcal{E}^+ \cap \mathcal{N}^+ \neq \emptyset$, $\lambda_*^+ > 1$ *and*
(H4) $(\mathcal{A}_0^+ \cap \mathcal{B}_0) \setminus \{0\} \subset \mathcal{E}^+$,
(H5) $\mathcal{A}^+ \subset \mathcal{E}^+$.
Then the following local infimum

$$i := \inf_{u \in \mathcal{N}^+ \cap \mathcal{E}^+} J(u)$$

is achieved. Furthermore $i < 0$.

Proof. The fact that $i < 0$ readily follows from Lemma 3.2(b). First we prove that $\mathcal{N}^+ \cap \mathcal{E}^+$ is bounded. Indeed $\mathcal{N}^+ \cap \mathcal{E}^+ \subset \mathcal{N}^+ \cap \mathcal{A}^+$ because of inequality (3.4) and, since $\mathcal{A}_0^+$ is a weakly closed cone, we have

that the condition (2.3) of Lemma 2.2 corresponds to hypothesis (**H4**), thus we conclude that $\mathcal{N}^+ \cap \mathcal{E}^+$ is bounded. Notice that in particular we have that $i > -\infty$. To prove that the infimum i is achieved take a minimizing sequence u_n. Since this sequence is bounded there exists some $u_0 \in X$ such that, up to subsequence, $u_n \rightharpoonup u_0$. By using (3.2) we can write

$$J(u_n) = \left(\frac{1}{p} - \frac{1}{q}\right) E(u_n) - \left(\frac{1}{r} - \frac{1}{q}\right) A(u_n) \geq - \left(\frac{1}{r} - \frac{1}{q}\right) A(u_n).$$

and letting $n \to +\infty$ we get $A(u_0) \geq -i(\frac{1}{r} - \frac{1}{q})^{-1} > 0$. Then it comes then from (**H5**) that $E(u_0) > 0$. We claim that u_n converges strongly to u_0 in X. Assume by contradiction that $u_n \not\to u_0$. We discuss two alternatives :
Alt. 1. $B(u_0) > 0$. Using the previous classification of the fibering maps, the graph of J_{u_0} is as the one in Fig. 2b so there exist $0 < t_1(u_0) < t_2(u_0)$ such that $t_1(u_0)u_0 \in \mathcal{N}^+ \cap \mathcal{E}^+$, $t_2(u_0)u_0 \in \mathcal{N}^- \cap \mathcal{E}^+$, J_{u_0} is increasing between $t_1(u_0)$ and $t_2(u_0)$ and decreasing elsewhere. Since we are assuming that $u_n \not\to u_0$ hence $u_0 \notin \mathcal{N}$ and therefore $t_1(u_0) \neq 1$. Let us distinguish two cases: (a) $1 \leq t_2(u_0)$ and (b) $1 > t_2(u_0)$. In case (a) we have

$$J(t_1(u_0)u_0) = J_{u_0}(t_1(u_0)) \leq J_{u_0}(1) < \liminf J_{u_n}(1) = \lim J(u_n) = i. \quad (3.5)$$

Moreover Lemma 2.5 and hypothesis (**H4**) imply that $\mathcal{N}_0 = \{0\}$. Thus (3.5) leads to a contradiction because $t_1(u_0)u_0 \in \mathcal{N}^+ \cap \mathcal{E}^+$. In case (b) using that

$$0 = J'_{u_0}(t_2(u_0)) < \liminf J'_{u_n}(t_2(u_0))$$

we conclude that $J'_{u_n}(t_2(u_0)) > 0$ for n large. Since 1 is a local minimum of J_{u_n} and the graph of J_{u_n} looks like the one of Fig. 2b then it must be $1 < t_2(u_0)$, a contradiction.
Alt. 2. $B(u_0) \leq 0$. The graph of J_u is as the one in Fig. 2e so there exists $0 < t(u_0)$ such that $t(u_0)u_0 \in \mathcal{N}^+ \cap \mathcal{E}^+$ and J_{u_0} has a global minimum in $t(u_0)$. If $t(u_0) \geq 1$ again we have (3.5), a contradiction. If $t(u_0) < 1$ we use that

$$0 = J'_{u_0}(t(u_0)) < \liminf J'_{u_n}(t(u_0)),$$

so then $J'_{u_n}(1) > 0$ for n large, which is also impossible because $u_n \in \mathcal{N}$. □

3.2. Minimizing J along $\mathcal{N}^-$

We now look for solutions of $J'(u) = 0$ in $\mathcal{N}^- \cap \mathcal{E}^+$.

Theorem 3.4. *Let us assume that $\mathcal{N}^- \cap \mathcal{E}^+ \neq \emptyset$, $\lambda_*^+ > 1$ and*
(H6) $\quad \mathcal{B}_0^+ \setminus \{0\} \subset E^+$,
Then the following infimum

$$j := \inf_{u \in \mathcal{N}^- \cap \mathcal{E}^+} J(u)$$

is > 0 and it is achieved.

Proof. We know from Lemma 3.2(a) that $j \geq 0$. Observe that any minimizing sequence is bounded because (**H6**) implies the hypothesis (2.3) of Lemma 2.2 and the result comes from (ii) of the aforementioned lemma. Assume first that $j > 0$. We claim that there is a strong convergent minimizing sequence for j. Assume by contradiction that we have a minimizing sequence $u_n \rightharpoonup u_0$ such that $A(u_n) \to A(u_0), B(u_n) \to B(u_0)$ but $u_n \not\to u_0$. If $u_0 = 0$ then Lemma

2.1 will imply that $u_n \to 0$, which is not the case we are assuming now. Thus $u_0 \neq 0$. We can also prove that $B(u_0) > 0$ by using the fact that $u_n \in \mathcal{E}^+$ and

$$\frac{q-r}{qr}B(u_n) = J(u_n) - \left(\frac{1}{p} - \frac{1}{r}\right)E(u_u) \geq J(u_n).$$

Then passing to the limit we get $B(u_0) > 0$. Consequently, (**H6**) implies that $E(u_0) > 0$. Now we distinguish two cases according to the sign of $A(u)$.
Alt. 1. $A(u_0) > 0$. In this case J_{u_0} and J_{u_n} look like Fig. 2b. If $u_n \not\to u_0$ then $t_2(u_0) \neq 1$. We also have that $t_2(u_0)u_0 \in \mathcal{N}^- \cap \mathcal{E}^+$ (we use here that $\lambda_*^+ > 1$ to assure that $t_2(u_0)u_0 \notin \mathcal{N}_0$, c.f. Lemma 2.5). Furthermore

$$0 = J'_{u_0}(t_2(u_0)) < \liminf J'_{u_n}(t_2(u_0)).$$

Thus $J'_{u_n}(t_2(u_0)) > 0$ for n large. Since $t_2(u_n) = 1$ hence $t_1(u_n) < t_2(u_0) < 1$ and we will have

$$j \leq J_{u_0}(t_2(u_0)) < \liminf J_{u_n}(t_2(u_0)) \leq \lim_{n\to+\infty} J_{u_n}(1) = j, \tag{3.6}$$

a contradiction.
Alt. 2. $A(u_0) \leq 0$. In this case J_{u_0} and J_{u_n} look like Fig. 2c. If $u_n \not\to u_0$ then $t(u_0) \neq 1$. We also have that $t(u_0)u_0 \in \mathcal{N}^- \cap \mathcal{E}^+$. Then again we have

$$j \leq J_{u_0}(t(u_0)) < \liminf J_{u_n}(t(u_0)) \leq \lim_{n\to+\infty} J_{u_n}(1) = j, \tag{3.7}$$

a contradiction.

Let us finally prove that $j > 0$. If this were not the case then for any minimizing sequence (which we know that will be bounded) we will have $J(u_n) \to 0$. Then, up to a subsequence, there exists $u_0 \in X$ such that $u_n \rightharpoonup u_0$, $A(u_n) \to A(u_0)$ and $B(u_n) \to B(u_0)$. From Lemma 3.2(a) we know that $\mathcal{N}^- \cap \mathcal{E}^+ \subset \mathcal{B}^+$ and hence the possibility $u_0 = 0$ is excluded from Lemma 2.3 and (**H6**) applied to $C = \mathcal{B}_0^+$, so (2.4) is satisfied. Besides by writing

$$0 = \lim_{n\to+\infty} J(u_n) = \lim_{n\to+\infty} \left(\frac{1}{p} - \frac{1}{r}\right)E(u_n) + \left(\frac{1}{r} - \frac{1}{q}\right)B(u_n)$$

it comes

$$\lim_{n\to+\infty} E(u_n) = \frac{p(q-r)}{q(p-r)}B(u_0). \tag{3.8}$$

Thus, if $B(u_0) = 0$ then $E(u_0) \leq \lim_{n\to+\infty} E(u_n) = 0$, a contradiction with (**H6**). Hence $B(u_0) > 0$. Furthermore, from (3.8) and

$$0 = \lim_{n\to+\infty} J(u_n) = \lim_{n\to+\infty} \left(\frac{1}{p} - \frac{1}{q}\right)E(u_n) - \left(\frac{1}{r} - \frac{1}{q}\right)A(u_n)$$

it comes $A(u_0) = \dfrac{r(q-p)}{q(p-r)}B(u_0) > 0$. We are going to rule out the following two alternatives: Alt. 1. $u_n \to u_0$. In this case $J(u_0) = 0$ and we will have $u_0 \in \mathcal{N}^- \cap \mathcal{E}^+$ (the possibility of $u_0 \in \mathcal{N}_0$ is ruled by the constraint $\lambda_*^+ > 1$). But, according to Lemma 3.2(a), $J(u_0) > 0$, a contradiction.
Alt. 2. $u_n \not\to u_0$. The maps J_{u_0} will look as in Fig. 2b and repeating the argument of Alt. 1 above we have (3.6) and we reach a contradiction. This conclude the proof of $j > 0$. □

4. Local minimizers of J restricted to the Nehari set and to $\mathcal{E}^-$

4.1. Minimizing J along $\mathcal{N}^+$

We first look for solutions of our problem in $\mathcal{N}^+ \cap \mathcal{A}^- \subset \mathcal{N}^+ \cap \mathcal{E}^-$.

Theorem 4.1. *Assume that* $\mathcal{N}^+ \cap \mathcal{A}^- \neq \emptyset$, $\lambda_*^- > 1$ *and*
*(**H7**)* $\mathcal{A}_0^- \cap \mathcal{B}_0 \setminus \{0\} \subset \mathcal{E}^+$
*(**H8**)* $\mathcal{A}_0 \cap \mathcal{B}^- \subset \mathcal{E}^+$.
Then the following infimum

$$l := \inf_{u \in \mathcal{N}^+ \cap \mathcal{A}^-} J(u)$$

is < 0 *and it is achieved.*

Proof. It comes from (**H7**) and Lemma 2.2(i) that $\mathcal{N}^+ \cap \mathcal{A}^-$ is bounded and by Lemma 3.2(a) that l is negative. Let us prove that this infimum is achieved. Let u_n be a minimizing sequence. Since this sequence is bounded, there exists u_0 such that $u_n \rightharpoonup u_0, A(u_n) \to A(u_0)$ and $B(u_n) \to B(u_0)$. Thus $A(u_0) \leq 0$, $B(u_0) \leq 0$ and $E(u_0) \leq \liminf E(u_n) \leq 0$. From $E(u_n) \leq 0$ and

$$J(u_n) = \left(\frac{1}{p} - \frac{1}{r}\right) E(u_n) + \left(\frac{1}{r} - \frac{1}{q}\right) B(u_n)$$

we infer that

$$B(u_n) \leq \frac{rq}{q-r} J(u_n)$$

and passing to the limit it comes that $B(u_0) < 0$. In particular $u_0 \neq 0$. Now, if $A(u_0) = 0$, using (**H8**) we will conclude that $E(u_0) > 0$, a contradiction. Then $A(u_0) < 0$ and J_{u_0} behaves either as in Fig. 2d if $E(u_0) = 0$ or as in Fig. 2f if $E(u_0) < 0$. Let us discuss this two alternatives:
<u>Alt. 1</u>. $E(u_0) = 0$. In this case $E(u_0) = 0 = \liminf E(u_n) = 0$ so $u_n \to u_0$ by (H3). Hence $J(u_0) = l$ and also $u_0 \in \mathcal{N}_0^+ \cap \mathcal{A}^-$. Since be in $\mathcal{N}_0$ implies that $E(u_0) = \frac{q-r}{q-p} A(u_0)$ by (3.4) and we know that $A(u_0) < 0 = E(u_0)$, then clearly $u_0 \notin \mathcal{N}_0$, so have $u_0 \in \mathcal{N}^+ \cap \mathcal{A}^-$ and we are done.
<u>Alt. 2</u>. $E(u_0) < 0$. In this case there exist two values $0 \leq t_1(u_0) < t_2(u_0)$ such that $t_2(u_0) > 0$ is a global minimum value for J_{u_0}, $t_2(u_0)u_0 \in \mathcal{N}^+ \cap \mathcal{A}^-$ and J_{u_0} is decreasing between $t_1(u_0)$ and $t_2(u_0)$, increasing elsewhere. Notice that $u_0 \notin \mathcal{N}_0$ because of condition $\lambda_*^- > 1$. Let us assume by contradiction that $u_n \not\to u_0$. Then

$$l \leq J(t_2(u_0)u_0) = J_{u_0}(t_2(u_0)) \leq J_{u_0}(1) < \liminf J_{u_n}(1) = \lim J(u_n) = l,$$

a contradiction. □

4.2. Minimizing J along $\mathcal{N}^-$

Finally we minimize along $\mathcal{N}^- \cap \mathcal{B}^- \subset \mathcal{N}^- \cap \mathcal{E}^-$.

Theorem 4.2. *Assume that* $\mathcal{N}^- \cap \mathcal{B}^- \neq \emptyset$, $\lambda_*^- > 1$, *(**H7**) and*
*(**H9**)* $\mathcal{A}_0 \cap \mathcal{B}_0^- \setminus \{0\} \subset \mathcal{E}^+$.

Then the infimum

$$k := \inf_{u \in \mathcal{N}^- \cap \mathcal{B}^-} J(u)$$

is achieved and it is positive.

Proof. By Lemma 3.2(a) the value $k \geq 0$. Since $\mathcal{N}^- \cap \mathcal{B}^- \subset \mathcal{A}^-$, Lemma 2.2(ii) and (**H7**) imply that any minimizing sequence is bounded. We claim that any minimizing sequence u_n possesses a convergence subsequence. Indeed, assume $u_n \rightharpoonup u_0$, $A(u_n) \to A(u_0) \leq 0$ and $B(u_n) \to B(u_0) \leq 0$ and that $u_n \not\to u_0$. Since $u_n \in \mathcal{N}^-$ we have $E(u_n) \leq \frac{q-r}{p-r} B(u_n) \leq 0$ so $E(u_0) \leq 0$. Let us assume first that $k \neq 0$. If $u_0 = 0$ then from Lemma 2.1 it comes that $u_n \to 0$ and therefore $0 = \lim J(u_n) = k$, a contradiction. Thus $u_0 \neq 0$. From hypothesis (**H7**) and (**H9**) we must have $u_0 \in \mathcal{A}^- \cap \mathcal{B}^-$. Hence J_{u_0} looks like Fig. 2f if $E(u_0) < 0$ or like Fig. 2d if $E(u_0) = 0$. Let us to discuss this two alternatives.
<u>Alt. 1</u>. $E(u_0) = 0$. As we have $E(u_0) = 0 = \liminf E(u_n)$ then $u_n \to u_0$. Hence, using that $u_n \in \mathcal{N}$ and passing to the limit we find $0 = E(u_0) = A(u_0) + B(u_0) \leq 0$ so it must be $A(u_0) = B(u_0) = 0$, a contradiction with (**H7**) or (**H9**). We have ruled out this alternative.
<u>Alt. 2</u>. $E(u_0) < 0$. Thus there exists $0 < t_1(u_0) < t_2(u_0)$ such that $t_1(u_0)u_0 \in \mathcal{N}^-$ and $t_1(u_n) = 1$. We have used here that $t_1(u_0)u_0 \notin \mathcal{N}_0$ because of the hypothesis $\lambda_*^- > 1$. Let us assume by contradiction that $u_n \not\to u_0$. Hence

$$0 = J'_{u_0}(t_1(u_0)) < \liminf J'_{u_n}(t_1(u_0))$$

which implies that $J'_{u_n}(t_1(u_0)) > 0$ for n large. Thus $t_1(u_0) < 1$ or $t_1(u_0) > t_2(u_n)$ because J_{u_n} looks line Fig. 2f. In the first case we have

$$k \leq J(t_1(u_0)u_0) = J_{u_0}(t_1(u_0)) < \liminf J_{u_n}(t_1(u_0)) < \liminf J_{u_n}(1) = k,$$

a contradiction. In the second case let us suppose, up to a subsequence, that $t_2(u_n)$ converges to some $s \in [1, t_1(u_0)]$. We have

$$J'_{u_0}(s)) < \liminf J'_{u_n}(t_2(u_n)) = 0,$$

which is a contradiction because $J'_{u_0} > 0$ on $(0, t_1(u_0))$. We have just proved that the minimizing sequence converges strongly and consequently the infimum k is achieved.

To finish the proof let us check that $k > 0$. Assume by contradiction that $k = 0$ and take u_n a minimizing sequence. Since u_n is bounded (as above) then we can assume that $u_n \rightharpoonup u_0$ for some u_0. We have $B(u_0) \leq 0$, $A(u_0) \leq 0$ and $u_0 \in \mathcal{E}_0^-$. It follows from Lemma 2.3 and (**H9**) that $u_0 \neq 0$ and from the fact that $u_n \in \mathcal{N}^-$ we have

$$A(u_n) > \frac{pqr}{(q-r)(r-p)} J(u_n).$$

Hence passing to the limit we get $A(u_0) = 0$. Thus $u_0 \in \mathcal{A}_0 \cap \mathcal{B}_0^- \cap \mathcal{E}_0^-$ which contradicts (**H9**) if $B(u_0) \neq 0$ or (**H7**) if $B(u_0) = 0$. □

5. On various conditions for coerciveness and existence results

5.1. On the coerciveness of E restricted to $\mathcal{A}_0^{\pm}$ and $\mathcal{B}_0^{\pm}$

We want to give in this section some variational conditions on E, A, B that would imply hypothesis (**H4**)–(**H9**). Those conditions will concern the following **constants of coerciveness**

$$\begin{aligned} i^{\pm}(A) &:= \inf\{E(u)\ ;\ A(u)=\pm 1\}, \\ i^{\pm}(B) &=: \inf\{E(u)\ ;\ B(u)=\pm 1\}, \end{aligned} \tag{5.1}$$

$$\begin{aligned} j_0(A) &:= \inf\{E(u)\ ;\ A(u)=0, u\in\mathcal{S}\}, \\ j_0(B) &:= \inf\{E(u)\ ;\ B(u)=0, u\in\mathcal{S}\}. \end{aligned} \tag{5.2}$$

Recall that $\mathcal{S}$ is the unit sphere of X. We give below and existence and multiplicity result for the equation $J'(u)=0$ in terms of the constants $j_0, i^{\pm}$.

Theorem 5.1. *Let us assume hypothesis (**H1**) to (**H3**).*

(i) *If $\lambda_*^+ > 1$, $\mathcal{N}^+ \cap \mathcal{E}^+ \neq \emptyset$, $i^+(A) > 0$ and either $j_0(A) > 0$ or $j_0(B) > 0$ then there exists at least one solution of $J'(u) = 0$ in $\mathcal{N}^+ \cap \mathcal{E}^+$.*

(ii) *If $\lambda_*^+ > 1$, $\mathcal{N}^- \cap \mathcal{E}^+ \neq \emptyset$, $j_0(B) > 0$ and $i^+(B) > 0$ then there exists at least one solution of $J'(u) = 0$ in $\mathcal{N}^- \cap \mathcal{E}^+$.*

(iii) *If $\lambda_*^- > 1$, $\mathcal{N}^+ \cap \mathcal{A}^- \neq \emptyset$ and either*
  (1) *$j_0(A) > 0$ and $i^-(A) > 0$ or*
  (2) *$j_0(B) > 0$ and $j_0(A) > 0$ or*
  (3) *$j_0(B) > 0$ and $i^-(B) > 0$,*
then there exists at least one solution of $J'(u) = 0$ in $\mathcal{N}^+ \cap \mathcal{A}^- \subset \mathcal{N}^+ \cap \mathcal{E}^-$.

(iv) *If $\lambda_*^- > 1$, $\mathcal{N}^- \cap \mathcal{B}^- \neq \emptyset$ and either*
  (4) *$j_0(A) > 0$ and $j_0(B) > 0$ or*
  (5) *$j_0(A) > 0$ and $i^-(A) > 0$ or*
  (6) *$j_0(B) > 0$ and $i^-(B) > 0$,*
then there exists at least one solution of $J'(u) = 0$ in $\mathcal{N}^- \cap \mathcal{B}^- \subset \mathcal{N}^- \cap \mathcal{E}^-$.

We left the proof to the reader. In the next proposition we give a variational characterization of $i^{\pm}(A)$ and $i^{\pm}(B)$.

Proposition 5.2. *Let us assume hypothesis (**H1**) to (**H3**). Assume that $j_0(A) > 0$ and that $i^{\pm}(A) \in \mathbb{R}$. Then there exits $\varphi_{\pm} \in X$ satisfying $A(\varphi_{\pm}) = \pm 1$ such that*

$$E'(\varphi_{\pm}) = \frac{p}{r} i^{\pm}(A) A'(\varphi_{\pm}).$$

Similarly, if $j_0(B) > 0$ and $i^{\pm}(B) \in \mathbb{R}$ then there exits $\phi_{\pm} \in X$ satisfying $B(\phi_{\pm}) = \pm 1$ such that

$$E'(\phi_{\pm}) = \frac{p}{q} i^{\pm}(B) B'(\phi_{\pm}).$$

Proof. We only prove the first part. If $u_n \in X$ with $A(u_n) = 1$ is a minimizing sequence for, say, $i^+(A)$ then the sequence u_n is bounded. Indeed, otherwise if we take $v_n := \frac{v_n}{\|u_n\|}$ hence, up to a subsequence, there exists $v_0 \in X$ such that $v_n \rightharpoonup v_0$, $A(v_n) \to 0$ and $E(v_n) \to 0$. Notice that $v_0 \neq 0$ because otherwise it will follows from $0 = E(v_0) \leq \liminf E(v_n) = 0$ and (**H3**) that $v_n \to 0$. This is impossible since $\|v_n\| = 1$. We then have $v_0 \neq 0$, $A(v_0) = 0$ and $E(v_0) \leq \liminf E(v_n) = 0$, a contradiction with the hypothesis $j_0(A) > 0$. We have just proved that u_n is bounded. The proof that $i^+(A)$ is achieved is standard and we omit it here. Let us denote by φ_+ an element of X where the infimum is achieved. From the Lagrange multipliers rule it follows that there exists $\lambda \in \mathbb{R}$ such that $E'(\varphi_+) = \lambda A'(\varphi_+)$. By testing this last equation against φ_+ and using the different homogeneities of E and A we get the result. $\square$

Remark 5.3. As a matter of fact the constraint $A(u) = 1$ in the definition of $i^+(A) := i^+(A, 1)$ can be replaced by, say, $A(u) = c$, where c in any positive number. In that case, it is trivial to prove that $i^+(A, c) = c^{\frac{p}{r}} i^+(A, 1)$. A similar statement can be formulated for $i^-(A)$ and $i^\pm(B)$.

Remark 5.4. We can easily prove that $j_0(A)$ and $j_0(B)$ are achieved provided they are finite and $j_0(A, B) := \inf\{E(u)\ ;\ A(u) = 0, B(u) = 0, u \in \mathcal{S}\} > 0$. However we can not give a variational formulation of any of them mainly because, in general, we don't know if the set $A^{-1}(\{0\})$ (resp. $B^{-1}(\{0\})$) is a manifold.

5.2. On the coerciveness of E and a principal eigenvalue

In this section we look for sufficient conditions implying (**H4**) to (**H9**) in terms of the *first eigenvalue* of the operator E. Precisely, let us assume the following

$$(\mathbf{H3})' \begin{cases} \exists (Y, \|\cdot\|_Y) \text{a Banach space s.t} X \text{ is cont. one-to-one embeded on} Y, \\ X \hookrightarrow Y \text{ is compact and } u \to \|u\|_Y^p \text{ is Fréchet differentiable.} \end{cases}$$

Let us denote $\chi(u) = \|u\|_Y^p$. Then it is straightforward from (**H2**)–(**H3**)' that

$$\lambda_1 := \inf\{E(u)\ ;\ \|u\|_Y = 1\} \tag{5.3}$$

is $> -\infty$ and it is achieved. Moreover, if $E(\varphi) = \lambda_1$ and $\|\varphi\|_Y = 1$ then, by Lagrange multiplier rule, $E'(\varphi) = \lambda_1 \chi'(\varphi)$.

We can give now the following sufficient conditions in terms of λ_1 to assure (**H4**)–(**H9**). Notice that the following conditions are stronger than the ones in terms of $i^\pm(A)$, $i^\pm(B)$ etc. Let us also stress here that $\lambda_1 > 0$ is equivalent to $\mathcal{E}_0^- \setminus \{0\} = \emptyset$ and therefore (**H4**) to (**H9**) are trivially satisfied. Hereafter we denote

$$E_{\lambda_1} := \{\varphi \in X \setminus \{0\}\ ;\ E'(\varphi) = \lambda_1 \chi'(\varphi)\}.$$

We have

Proposition 5.5. *Let us assume (**H1**) to (**H3**)′ and $\lambda_1 = 0$. Then*

(a) $E_{\lambda_1} \cap \mathcal{A}_0^+ \cap \mathcal{B}_0 = \emptyset \Rightarrow$ (***H4***),
(b) $E_{\lambda_1} \subset \mathcal{A}^- \Rightarrow j_0(A) > 0, i^+(A) > 0 \Rightarrow j_0(A) > 0$ *and* (***H5***),
(c) $E_{\lambda_1} \subset \mathcal{B}^- \Rightarrow j_0(B) > 0, i^+(B) > 0 \Rightarrow$ (***H6***).

Proof. We only prove (b), the proof of the other cases are similar. Trivially $i^+(A) \geq \lambda_1 = 0$ and $j_0(A) \geq \lambda_1 = 0$. Assume by contradiction that $i^+(A) = 0$ and let $u_n \in X$ with $A(u_n) = 1$ be a minimizing sequence for $i^+(A)$. If the sequence u_n is bounded then, up to a subsequence, $u_n \rightharpoonup u_0$ for some $u_0 \in X$. Hence $A(u_0) = 1$ (so in particular $u_0 \not\equiv 0$) and $E(u_0) = i^+(A) = 0$. Hence

$0 = \lambda_1 = E\left(\frac{u_0}{\|u_0\|_Y}\right)$ so $\frac{u_0}{\|u_0\|_Y}$ is an eigenfunction associated to λ_1 and we have again a contradiction with the assumption of (b). Thus the sequence (u_n) is unbounded. Let us take $v_n := \frac{v_n}{\|u_n\|}$; hence there exists $v_0 \in X$ such that, up to a subsequence, $v_n \rightharpoonup v_0$, $A(v_n) \to 0$ and $E(v_n) \to 0$. Notice that $v_0 \neq 0$ because otherwise it will follows from $0 = E(v_0) \leq \liminf E(v_n) = 0$ and (**H3**) that $v_n \to 0$. This is impossible since $\|v_n\| = 1$. We then have $v_0 \neq 0$, $A(v_0) = 0$ and $E(v_0) \leq 0$. By using the inequality

$$0 = \lambda_1 \leq E\left(\frac{v_0}{\|v_0\|_Y}\right)$$

we deduce that $E(v_0) = 0$ so v_0 belongs to E_{λ_1}, a contradiction with the assumption of (b). The proof of $j_0(A) > 0$ rules similarly. □

We can obtain two more solutions of $J'(u) = 0$ in the case $\lambda_1 < 0$, that is, when $\mathcal{E}^- \neq \emptyset$. To do so, let us assume in this case that

(**H3**)″ $\sigma_Y(E) := \{\lambda \in \mathbb{R}\ ;\ \exists u \in X \setminus \{0\}, E'(u) = \lambda \chi'(u)\} \subset \{\lambda_1\} \cup]0, +\infty[$.

We have

Proposition 5.6. *Let us assume hypothesis (**H1**) to (**H3**). Then*

$$\begin{array}{ll} (a) & j_0(A) > 0 \Rightarrow (\textbf{\textit{H8}}) \text{ and } (\textbf{\textit{H9}}), \\ (b) & j_0(B) > 0 \Rightarrow (\textbf{\textit{H4}}) \text{ and } (\textbf{\textit{H7}}). \end{array}$$

*Let us assume also hypothesis (**H3**)′ and (**H3**)″. Then*

(c) $j_0(A) > 0$ *and* $E_{\lambda_1} \subset \mathcal{A}^- \Rightarrow$ (***H4***) *and* $i^+(A) > 0 \Rightarrow$ (***H4***) *and* (***H5***),
(d) $j_0(B) > 0$ *and* $E_{\lambda_1} \subset \mathcal{B}^- \Rightarrow$ (***H4***) *and* $i^+(B) > 0 \Rightarrow$ (***H4***) *and* (***H6***).

Proof. (a), (b) are trivial. We only prove (c) as the proof of (d) is similar. Let us denote

$$d := \inf\{E(u);\ A(u) \geq 0, \|u\|_Y = 1\}$$

and prove that $d > 0$. Clearly $d > 0 \Rightarrow$ (**H4**). Since for all $u \in X$ satisfying $A(u) = 1$ we have $E(\frac{u}{\|u\|_Y}) \geq d$, the conclusion $i^+(A) > 0$ will follow, and also (**H5**).

First we claim that d is achieved. Indeed, if u_n is an admissible sequence with $E(u_n) \to d$ then we can prove that the sequence is bounded (otherwise the sequence $v_n = \frac{u_n}{\|u_n\|}$ will satisfy $E(v_n) \to 0, \|v_n\|_Y \to 0$, so, up to a subsequence, $v_n \rightharpoonup 0$, in contradiction with (**H3**) and the fact that $\|v_n\| = 1$). Let $u_0 \in X$ be such that $u_n \rightharpoonup u_0$. Thus $A(u_0) \geq 0, \|u_0\|_Y = 1$ and consequently $d \leq E(u_0) \leq \liminf E(u_n) = d$. We have proved that $E(u_0) = d$ and d is achieved. If $A(u_0) = 0$ hence $d \geq j_0(A) > 0$ and we are done. If $A(u_0) > 0$ hence

$$d = \inf\{E(u);\ A(u) > 0, \|u\|_Y = 1\}$$

and d is then a local minima of E under the constrain $\|u\|_Y = 1$. By Lagrange multiplier rule, there exists $\lambda \in \mathbb{R}$ such that $E'(u_0) = \lambda\chi'(u_0)$. By evaluating this equation at u_0 we readily obtain

$$pd = pE(u_0) = \langle E'(u_0), u_0\rangle = \lambda\langle \chi'(u_0), u_0\rangle = p\lambda.$$

Since $E_{\lambda_1} \subset \mathcal{A}^-$ then $\lambda \neq \lambda_1$ and therefore $d = \lambda > 0$ by $(\mathbf{H3})''$. □

We can now give the first main existence result of this section. This result generalizes Theorems 4.8 and 5.7 of [16].

Theorem 5.7. *Let us assume hypotheses* ($\boldsymbol{H1}$) *to* $(\boldsymbol{H3})'$.

(i) *Assume that* $\mathcal{N}^\pm \cap \mathcal{E}^+ \neq \emptyset$ *and* $\lambda_*^+ > 1$. *If either* $\lambda_1 > 0$ *or* $\lambda_1 = 0$ *and* $E_{\lambda_1} \subset \mathcal{A}^- \cap \mathcal{B}^-$ *then there exists at least two solutions of* $J'(u) = 0$.

(ii) *Assume that* $\mathcal{N}^+ \cap \mathcal{E}^+ \neq \emptyset$ *and* $\lambda_*^+ > 1$. *If* $\lambda_1 = 0$ *and* $E_{\lambda_1} \subset \mathcal{A}^- \cap \mathcal{B}_0^+$ *then there exists at least one solution of* $J'(u) = 0$.

(iii) *Assume that* $\mathcal{N}^+ \cap \mathcal{A}^- \neq \emptyset$, $\mathcal{N}^- \cap \mathcal{B}^- \neq \emptyset$ *and* $\lambda_*^- > 1$. *If* $j_0(A) > 0$ *and* $j_0(B) > 0$ *then there exist at least two solutions of* $J'(u) = 0$ *in* $\mathcal{E}^-$.

(iv) *Assume* $(\boldsymbol{H3})''$, $\mathcal{N}^+ \cap \mathcal{E}^+ \neq \emptyset$, $\lambda_*^+ > 1$ *and* $\lambda_1 < 0$. *If* $j_0(A) > 0$ *and* $E_{\lambda_1} \subset \mathcal{A}^-$ *then there exist at least one solution of* $J'(u) = 0$ *in* $\mathcal{E}^+$.

(v) *Assume* $(\boldsymbol{H3})''$, $\mathcal{N}^\pm \cap \mathcal{E}^+ \neq \emptyset$, $\lambda_*^+ > 1$ *and* $\lambda_1 < 0$. *If* $j_0(A) > 0$, $j_0(B) > 0$ *and* $E_{\lambda_1} \subset \mathcal{A}^- \cap \mathcal{B}^-$ *then there exist at least two solutions of* $J'(u) = 0$ *in* $\mathcal{E}^+$.

Proof. (i) Clearly we have (**H4**), (**H5**) and (**H6**) from Proposition 5.5. Thus the local minimum of Theorem 3.3 provides us a first solution of $J'(u) = 0$. Since we also have (**H6**), a second solution comes from Theorem 3.4.

(ii) We have (**H4**) and (**H5**) from Proposition 5.5 (a)–(b). Thus the local minimum of Theorem 3.3 is a solution of $J'(u) = 0$.

(iii) From (a) and (b) of Proposition 5.6 the hypotheses (**H7**), (**H8**) and (**H9**) hold. Then we get two solutions from Theorems 4.1 and 4.2.

(iv) From (c) of Proposition 5.6 the hypotheses (**H4**) and (**H5**) hold. Then we get one solution from Theorem 3.3.

(v) From (c) and (d) of Proposition 5.6 the hypotheses (**H4**), (**H5**), (**H6**) and (**H7**) hold. Then we get two solutions from Theorems 3.3 and 3.4. □

5.3. A variational characterization of $\boldsymbol{\lambda_*^\pm}$

We have proved in the previous section the existence of four solutions of the equation $J'(u) = 0$ provided $\lambda_*^\pm > 1$, where $\lambda_*^\pm$ has been defined in (2.8) and (2.9). Let us here give some variational characterization of these values. For the sake of simplicity let us denote

$$\alpha := \frac{p-r}{q-r}, \quad \beta := \frac{q-p}{q-r},$$

so it holds

$$0 < \alpha, \beta < 1, \quad \alpha + \beta = 1, \quad q\alpha + r\beta = p.$$

Observe that

$$(\lambda_*^+)^\beta = \beta^\beta\alpha^\alpha \inf\{E(u);\ u \in \mathcal{E}^+ \cap \mathcal{A}^+ \cap \mathcal{B}^+,\ A(u)^\beta B(u)^\alpha = 1\},$$
$$(\lambda_*^-)^\beta = \beta^\beta\alpha^\alpha \inf\{-E(u);\ u \in \mathcal{E}^- \cap \mathcal{A}^- \cap \mathcal{B}^-,\ (-A(u))^\beta(-B(u))^\alpha = 1\}.$$

Proposition 5.8. *Let us define*

$$j_0^+(A,B) := \inf\left\{E(u)\ ;\ u \in \mathcal{A}_0^+ \cap \mathcal{B}_0^+ \cap \mathcal{S}\right\} \tag{5.4}$$
$$j_0^-(A,B) := \inf\left\{E(u)\ ;\ u \in \mathcal{A}_0^- \cap \mathcal{B}_0^- \cap \mathcal{S}\right\}. \tag{5.5}$$

If $j_0^+(A,B) > 0$ then λ_*^+ is achieved. Furthermore, for any $u \in X$ where λ_*^+ is achieved we have

$$E'(u) = \frac{(\beta/\alpha)^\alpha(\lambda_*^+)^\beta}{A(u)}A'(u) + \frac{(\alpha/\beta)^\beta(\lambda_*^+)^\beta}{B(u)}B'(u) \tag{5.6}$$

A similar result holds for λ_*^- under the constraint $j_0^-(A,B) > 0$.

Proof. If $u_n \in \mathcal{A}^+ \cap \mathcal{B}^+ \cap \mathcal{E}^+$ with $A(u_n)^\beta B(u_n)^\alpha = 1$ is a minimizing sequence for λ_*^+ and (u_n) is bounded then, up to a subsequence, $u_n \rightharpoonup u_0$ for some $u_0 \in X$ that will satisfy $A(u_0)^\beta B(u_0)^\alpha = 1$, so u_0 will be admissible in the infimum and then it is achieved. If $\|u_n\|$ goes to $+\infty$ then for $v_n = \frac{u_n}{\|u_n\|}$ we will have, up to a subsequence, $v_n \rightharpoonup v_0$ for some $v_0 \in X$, $A(v_n)^\beta B(v_n)^\alpha = \frac{1}{\|u_n\|^p} \to 0$ and $E(v_0) \leq \liminf \frac{E(u_n)}{\|u_n\|^p} = 0$. We can rule out the possibility $v_0 = 0$, since in this case $E(v_0) = 0 = \liminf E(v_n)$ and then, from (**H3**), $v_n \to v_0$ which is impossible because $\|v_n\| = 1$. Thus $v_0 \in \mathcal{E}_0^- \cap \mathcal{A}_0^+ \cap \mathcal{B}_0^+$, a contradiction with the constraint $j_0^+(A,B) > 0$. Let us denote $\Phi(v) := A(v)^\beta B(v)^\alpha$ and observe that $\Phi^{-1}\{1\}$ is a C^1-manifold. By Lagrange Multiplier rule we have that for any $u \in \Phi^{-1}(1)$ with $E(u) = \frac{(\lambda_*^+)^\beta}{\alpha^\alpha\beta^\beta}$ there exits $\mu \in \mathbb{R}$ such that $E'(u) = \mu\Phi'(u)$. Testing this identity at u we have

$$\langle E'(u), u\rangle = \mu\big(\beta B(u)^\alpha A(u)^{\beta-1}\langle A'(u), u\rangle + \alpha A(u)^\beta B(u)^{\alpha-1}\langle B'(u), u\rangle\big)$$

that is $pE(u) = \mu(r\beta + q\alpha) = \mu p$ and the identity (5.6) follows after a simple computation. □

Remark 5.9. As in Proposition 5.5 one has

(i) $\lambda_1 > 0 \Rightarrow j_0^\pm(A,B) > 0$,

(ii) If $\lambda_1 = 0$ then

$$(1)\ E_{\lambda_1} \cap \mathcal{A}_0^+ \cap \mathcal{B}_0^+ = \emptyset \Rightarrow j_0^+(A,B) > 0.$$
$$(2)\ E_{\lambda_1} \cap \mathcal{A}_0^- \cap \mathcal{B}_0^- = \emptyset \Rightarrow j_0^-(A,B) > 0.$$

Let us show that, in a particular case, the variational equation (5.6) is equivalent to the equation $J'(u) = 0$. By *equivalent* we mean here that a multiple of a solution of (5.6) is a solution of $J'(u)$. Let us observe that if $\varphi \in X \setminus \{0\}$ is a solution of (5.6) and we denote for simplicity

$$d_1 := \frac{(\beta/\alpha)^\alpha(\lambda_*^+)^\beta}{A(\varphi)}, \quad d_2 := \frac{(\alpha/\beta)^\beta(\lambda_*^+)^\beta}{B(\varphi)} \tag{5.7}$$

then for any $c > 0$ the function $v = c\varphi$ satisfies

$$E'(v) = d_1c^{p-r}A'(v) + d_2c^{p-q}B'(v).$$

Thus we have the following result

Proposition 5.10. *Assume that $j_0^+(A,B)>0$ and that*

$$(\lambda_*^+)^\beta = \left(\frac{p}{r}\right)^\beta . \left(\frac{p}{q}\right)^\alpha . \tag{5.8}$$

Then there exists a solution of $J'(u)=0$ satisfying $u\in\mathcal{A}^+\cap\mathcal{B}^+$. The constant $c_{pqr} := (\frac{p}{r})^\beta.(\frac{p}{q})^\alpha$ is >1.

A similar result can be state for λ_*^+ if $j_0^-(A,B)>0$.

Proof. Recall that $J'(u)=0$ if and only if $E'(u)=\frac{p}{r}A'(u)+\frac{p}{q}B'(u)$. Let $\varphi\in X$ be a point where $\lambda *^+$ is achieved. A simple computation shows that we can choose $c>0$ such that $c\varphi$ is a critical point of J, that is, $d_1c^{p-r}=\frac{p}{r}$ and $d_2c^{p-q}=\frac{p}{q}$ if and only if

$$\left(\frac{\frac{p}{r}}{d_1}\right)^\beta . \left(\frac{\frac{p}{q}}{d_2}\right)^\alpha = 1$$

and using the fact that $\alpha+\beta=1$, $A(\varphi)^\beta B(\varphi)^\alpha=1$ and (5.7) we get (5.8). Let us write

$$(c_{pqr})^{\frac{q-r}{r}} = \left(\frac{p}{r}\right)^{\frac{q}{r}-1} . \left(\frac{q}{r}\right)^{1-\frac{p}{r}} .$$

One has

$$\frac{q-r}{r}\ln(c_{pqr}) = (\frac{q}{r}-1)\ln\frac{p}{r} + (1-\frac{p}{r})\ln\frac{q}{r} = (x-1)\ln y-(y-1)\ln x,$$

where we have put $x:=\frac{q}{r}$ and $y:=\frac{p}{r}$. Using the fact that the function $f(z):=\frac{\ln z}{z-1}$ is strictly decreasing for $z>1$ we conclude that $\frac{q-r}{r}.\ln(c_{pqr})>0$ and therefore the constant $c_{pqr}>1$ as claimed. □

Remark 5.11. Notice that, if (5.8) holds, we can not distinguish the solution of the problem $J'(u)=0$ obtained in Proposition 5.10 from the one obtained in Theorem 3.3 or the one in Theorem 3.4.

6. Existence results for quasilinear problems

6.1. Existence and multiplicity results for Problem I

Let $\Omega\subset\mathbb{R}^N$ be a bounded smooth domain of class $C^{2,\alpha}$ $(0<\alpha<1)$ with outward unit normal ν on the boundary $\partial\Omega$. Let $V\in L^\infty(\Omega)$ and $(a,b)\in(C^s(\partial\Omega))^2$, for some $s\in(0,1)$, and allow them to change sign. The exponents r,q to satisfy $1<r<p<q<p_*$ where $p_*=\frac{p(N-1)}{(N-p)^+}$ the critical exponent for the trace embedding; ρ denotes the restriction to $\partial\Omega$ of the $(N-1)$-Hausdorff measure, which coincides with the usual Lebesgue surface measure as $\partial\Omega$ is regular enough. Finally let the number λ be a positive parameter.

Take $X=W^{1,p}(\Omega)$ with the usual Sobolev norm $\|\cdot\|_{1,p}$. Solutions of Problem I are understood in the weak sense, that is,

$$\int_\Omega \left(|\nabla u|^{p-2}\nabla u\nabla\varphi + V(x)|u|^{p-2}u\varphi\right) = \int_{\partial\Omega}\left(\lambda a|u|^{r-2}u + b|u|^{q-2}u\right)\varphi\, d\rho. \tag{6.1}$$

$\forall \varphi \in W^{1,p}(\Omega)$. Let us consider

$$E(u) = \int_\Omega (|\nabla u|^p + V(x)|u|^p)\,dx, \quad A(u) = \lambda \int_{\partial\Omega} a(\rho)|u|^r\,d\rho,$$
$$B(u) = \int_{\partial\Omega} b(\rho)|u|^q\,d\rho,$$

and the energy functional

$$J(u) = \frac{1}{p}E(u) - \frac{1}{r}A(u) - \frac{1}{q}B(u)$$
$$= \frac{1}{p}\int_\Omega (|\nabla u|^p + V(x)|u|^p)\;dx - \frac{\lambda}{r}\int_{\partial\Omega} a(\rho)|u|^r\,d\rho - \frac{1}{q}\int_{\partial\Omega} b(\rho)|u|^q\,d\rho.$$

It is clear that solutions of Problem I are positive critical points of J. It is also clear that A, B satisfy (2.1) and hypothesis (**H1**) since the trace operators $W^{1,p}(\Omega) \to L^r(\partial\Omega,\rho)$ and $W^{1,p}(\Omega) \to L^q(\partial\Omega,\rho)$ are compact (remember that $r, q < p_*$). Hypotheses (**H2**) and (**H3**) are well known properties of the $p-$laplacian operator, c.f. [18].

Let us consider the compact embedding $W^{1,p}(\Omega) \hookrightarrow Y = L^p(\Omega)$ and denote $\|\cdot\|_p$ the Lebesgue norm of $L^p(\Omega)$. The eigenvalue λ_1 defined in (5.3) takes the following expression:

$$\lambda_1 := \inf\left\{\int_\Omega (|\nabla u|^p + V(x)|u|^p)\,dx\;;\; u \in W^{1,p}(\Omega), \|u\|_p = 1\right\}$$

and it corresponds to the least eigenvalue μ of the following eigenvalue problem with *Newman boundary conditions* :

$$\begin{cases} -\Delta_p u + V(x)|u|^{p-2}u = \mu|u|^{p-1}u & \text{in } \Omega, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.2}$$

This problem should be understood in the weak sense, that is, for all $\varphi \in W^{1,p}(\Omega)$

$$\int_\Omega \left(|\nabla u|^{p-2}\nabla u \nabla\varphi + V(x)|u|^{p-2}u\varphi\right)\,dx = \mu\int_\Omega |u|^{p-2}u\varphi\,dx. \tag{6.3}$$

It is known (c.f. [6]) that λ_1 is simple and isolated in the sense that

$$\inf\left\{\mu > \lambda_1\;;\;\mu \text{ solves (6.3) for some } u \in W^{1,p}(\Omega)\setminus\{0\}\right\} > \lambda_1.$$

We then have that hypothesis (**H3**)' is satisfied and that E_{λ_1} is of dimension 1. We will denote by φ_1 the unique eigenfunction of L^p-norm equal to 1 associated to λ_1. It is also known that φ_1 is sign definite and never vanishes in $\overline{\Omega}$. Furthermore, there is a *second eigenvalue* λ_2 for problem (6.2) and it can be characterized as

$$\lambda_2 = \inf\left\{\mu > \lambda_1\;;\;\mu \text{ solves (6.2) for some } u \in W^{1,p}(\Omega)\setminus\{0\}\right\}. \tag{6.4}$$

In order to apply Theorem 5.7 we are going first to determinate under which conditions on a, b, V the Nehari sign-sets used in Theorems 3.3, 3.4, 4.2, 4.1 are non empty. For this purpose we will use Proposition 2.4 of Sect. 2. Let us denote

$$\Gamma_a^{\pm} := \{\rho \in \partial\Omega \; ; \; a(\rho) \gtrless 0\}, \quad \Gamma_b^{\pm} := \{\rho \in \partial\Omega \; ; \; b(\rho) \gtrless 0\},$$
$$\Gamma_{a,0} := \{\rho \in \partial\Omega \; ; \; a(\rho) = 0\}, \quad \Gamma_{b,0} := \{\rho \in \partial\Omega \; ; \; b(\rho) = 0\}.$$

Lemma 6.1. (1) *$\mathcal{E}^+ \neq \emptyset$.*
(2) *If $\Gamma_a^+ \neq \emptyset$ then $\mathcal{N}^+ \cap \mathcal{E}^+ \neq \emptyset$.*
(3) *If $\Gamma_b^+ \neq \emptyset$ then $\mathcal{N}^- \cap \mathcal{E}^+ \neq \emptyset$.*
(4) *If $\Gamma_a^- \cap \Gamma_b^- \neq \emptyset$ then $\mathcal{N}^+ \cap \mathcal{A}^- \neq \emptyset$ and $\mathcal{N}^- \cap \mathcal{B}^- \neq \emptyset$.*

Proof. The proof of (1) is trivial and we only prove (2), the proofs of the other cases are similar.
(2) We distinguish two cases:
Case a) $\Gamma_a^+ \cap (\Gamma_b^- \cup \Gamma_{b,0}) \neq \emptyset$. In this case we can construct a C^∞ function v in $\partial\Omega$ such that $A(v) > 0 \geq B(v)$. Let $u \in W^{1,p}(\Omega)$ having v as its trace. Let ξ be a C^∞ cut function such that $0 \leq \xi \leq 1$, $\xi \equiv 1$ in a small ball $B(x_0, r) \subset \Omega$ where $|u| > c$ for some $c > 0$, $\xi \equiv 1$ in a neighbourhood $\Omega_\delta := \{x \in \overline{\Omega} \; ; \; dist\,(x, \partial\Omega) < \delta\}$ of $\partial\Omega$, and $\xi = 0$ in $\Omega \setminus \big(B(x, 2r) \cup \Omega_{2\delta}\big)$. We can assume that

$$c^p \int_{B(x_0,2r)} |\nabla\xi|^p \geq \int_{B(x_0,2r)} |\xi|^p \left(|\nabla u|^p + V|u|^p\right) \tag{6.5}$$

which implies that $E(\xi u) > 0$. Thus $\xi u \in \mathcal{A}^+ \cap \mathcal{B}_0^- \cap \mathcal{E}^+$ and from Proposition 2.4 (i) we infer that $\mathcal{N}^+ \cap \mathcal{E}^+ \neq \emptyset$.
Case b) If $\Gamma_a^+ \subset \Gamma_b^+$ the construction of u and ξ runs similarly, starting with $v \in C^\infty(\partial\Omega)$ such that $A(v), B(v) > 0$. The cut function ξ can be chosen in such a way that (6.5) is satisfied as well as

$$A(\xi u) = A(v) < \left(\frac{q-p}{q-r}\right)\left(\frac{p-r}{q-r}\right)^{\frac{p-r}{q-p}} \frac{E(\xi u)^{\frac{q-r}{q-p}}}{B(\xi u)^{\frac{p-r}{q-p}}} = \max_{t>0} m_{\xi u}(t).$$

Thus $\xi u \in \Lambda^+$ and the conclusion follows from (i) of Proposition 2.4. □

We remember here the coerciveness values defined in (5.1), that in our case will be

$$j_0(a) := \inf\left\{\int_\Omega |\nabla u|^p + V(x)|u|^p \; ; \; \int_{\partial\Omega} a|u|^r = 0, \|u\|_{1,p} = 1\right\},$$
$$j_0(b) := \inf\left\{\int_\Omega |\nabla u|^p + V(x)|u|^p \; ; \; \int_{\partial\Omega} b|u|^q = 0, \|u\|_{1,p} = 1\right\}.$$

Finally let us recall the definitions (2.8) and (2.9) and write for sake of simplicity

$$\mu_+ := \inf_{u \in \mathcal{A}^+ \cap \mathcal{B}^+ \cap \mathcal{E}^+} \frac{\max\limits_{t>0} m_u(t)}{\frac{A(u)}{\lambda}}$$

and

$$\mu_- := \inf_{u\in\mathcal{A}^-\cap\mathcal{B}^-\cap\mathcal{E}^-} \frac{\min_{t>0} m_u(t)}{\frac{A(u)}{\lambda}}$$

which do not depend on λ.

We can now reformulate Theorem 5.7 as the following existence and multiplicity result for Problem I.

Theorem 6.2. (i) *Assume that* $\Gamma_a^+ \neq \emptyset$ *and* $\Gamma_b^+ \neq \emptyset$. *If either* $\lambda_1 > 0$ *or* $\lambda_1 = 0$ *and* $\int_{\partial\Omega} a\varphi_1^r < 0$, $\int_{\partial\Omega} b\varphi_1^q < 0$ *then there exists at least two solutions for any* $\lambda \in (0, \mu_+^{-1})$.

(ii) *Assume that* $\Gamma_a^+ \neq \emptyset$. *If* $\lambda_1 = 0$ *and* $\int_{\partial\Omega} a\varphi_1^r < 0$ *then there exists at least one solution for any* $\lambda \in (0, \mu_+^{-1})$.

(iii) *Assume that* $\Gamma_a^- \cap \Gamma_b^- \neq \emptyset$. *If* $j_0(a) > 0$ *and* $j_0(b) > 0$ *then there exist at least two solutions in* $\mathcal{E}^-$ *for any* $\lambda \in (0, \mu_-^{-1})$.

(iv) *Assume that* $\Gamma_a^+ \neq \emptyset$. *If* $\lambda_1 < 0 < \lambda_2$, $j_0(a) > 0$ *and* $\int_{\partial\Omega} a\varphi_1^r < 0$ *then there exist at least one solution in* $\mathcal{E}^+$ *for any* $\lambda \in (0, \mu_+^{-1})$

(v) *Assume that* $\Gamma_a^+ \neq \emptyset$ *and* $\Gamma_b^+ \neq \emptyset$. *If* $\lambda_1 < 0 < \lambda_2$, $j_0(a) > 0$, $j_0(b) > 0$, $\int_{\partial\Omega} a\varphi_1^r < 0$, *and* $\int_{\partial\Omega} b\varphi_1^q < 0$ *then there exist at least two solutions in* $\mathcal{E}^+$ *for any* $\lambda \in (0, \mu_+^{-1})$.

In particular we have from cases (iii) and (v) that

Corollary 6.3. *Assume that* $\Gamma_a^+ \neq \emptyset$, $\Gamma_b^+ \neq \emptyset$ *and* $\Gamma_a^- \cap \Gamma_b^- \neq \emptyset$. *If* $\lambda_1 < 0 < \lambda_2$, $j_0(a) > 0$, $j_0(b) > 0$, $\int_{\partial\Omega} a\varphi_1^r < 0$, *and* $\int_{\partial\Omega} b\varphi_1^q < 0$ *then Problem I possesses* **at least 4 solutions** *for any* $\lambda \in (0, \min\{\mu_+^{-1}, \mu_-^{-1}\})$.

Proof of Theorem 6.2. The existence of weak solutions in each of the 4 cases have already be done in Theorem 5.7. Since each solution comes as a local minimizer of J along the sign subsets of the Nehari set and all of these subsets are invariant by taking the absolute value of a function u, we can assume that all these critical points are ≥ 0. Besides the result of [6, Theorem A.1] implies that all solutions are bounded and the regularity result of [14] gives that they are of class $C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$. Finally the Strong Maximum principle of [20] insures that non negative solutions of the problem are > 0 on $\overline{\Omega}$. □

Remark 6.4. Unfortunately we are not able to replace the conditions $j_0(a) > 0$ and $j_0(b) > 0$ by, say, a condition related to some suitable eigenvalue. We just remark that both $j_0(a) > 0$ and $j_0(b) > 0$ imply that $\lambda_1^D > 0$, where λ_1^D denotes the first eigenvalue of $-\Delta_p + V|u|^{p-2}u$ with *Dirichlet boundary conditions.*

Remark 6.5. In the case $a \geq 0$ (resp. $b \geq 0$) and a "nice" zero set $\Gamma_{a,0}$ one should be able to relate the condition $j_0(a) > 0$ (resp $j_0(b) > 0$) with the positivity of the first eigenvalue of $-\Delta_p u + V|u|^{p-2}u$ over $W^{1,p}(\Omega, \Gamma_{a,0}) := \{u \in W^{1,p}(\Omega) \ ; \ u = 0 \text{ on } \partial\Omega \setminus \Gamma_{a,0}\}$, as was done in [7, Proposition 11].

6.2. Existence and multiplicity results for Problem II

Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain of class $C^{2,\alpha}$ $(0 < \alpha < 1)$ with outward unit normal ν on the boundary $\partial\Omega$. Let $a, V \in L^\infty(\Omega)$ and $b \in C^s(\partial\Omega)$ for some $s \in (0,1)$ be possibly indefinite and the exponents r, q to satisfy $1 < r < p < q < p_*$ where $p_* = \frac{p(N-1)}{(N-p)^+}$ and let the number λ be a positive parameter. Take $X = W^{1,p}(\Omega)$ with the usual Sobolev norm $\|\cdot\|_{1,p}$, the operators

$$E(u) = \int_\Omega (|\nabla u|^p + V(x)|u|^p)\, dx, \quad A(u) = \lambda \int_\Omega a(x)|u|^r\, dx,$$
$$B(u) = \int_{\partial\Omega} b(\rho)|u|^q\, d\rho,$$

and the energy functional

$$\begin{aligned} J(u) &= \frac{1}{p}E(u) - \frac{1}{r}A(u) - \frac{1}{q}B(u) \\ &= \frac{1}{p}\int_\Omega (|\nabla u|^p + V(x)|u|^p)\, dx - \frac{\lambda}{r}\int_\Omega a(x)|u|^r\, dx - \frac{1}{q}\int_{\partial\Omega} b(\rho)|u|^q\, d\rho. \end{aligned}$$

A, B satisfy (2.1) and hypothesis (**H1**) since the embedding $W^{1,p}(\Omega) \hookrightarrow L^r(\Omega)$ and the trace operator $W^{1,p}(\Omega) \to L^q(\partial\Omega, \rho)$ are compact (remember that $r, q < p_* < p^*$). Solutions of Problem II are understood in the weak sense, that is,

$$\int_\Omega \left(|\nabla u|^{p-2}\nabla u \nabla\varphi + V(x)|u|^{p-2}u\varphi\right) = \int_\Omega \lambda a|u|^{r-2}u\varphi\, dx + \int_{\partial\Omega} b|u|^{q-2}u\varphi\, d\rho, \tag{6.6}$$

for all $\varphi \in W^{1,p}(\Omega)$. Similar to the previous case, (**H2**) to (**H3**)" are also satisfied with λ_1 defined in (6.2) and λ_2 as in (6.4). Let us consider again the compact embedding $W^{1,p}(\Omega) \hookrightarrow Y = L^p(\Omega)$ so the eigenvalues λ_1 and λ_2 have been already defined for Problem I. Let us denote

$$\Omega_a^\pm := \{x \in \Omega\ ;\ a(x) \gtrless 0\}, \quad \Gamma_b^\pm := \{\rho \in \partial\Omega\ ;\ b(\rho) \gtrless 0\}.$$

We study now the sign-sets associated to J.

Lemma 6.6. (1) $\mathcal{E}^+ \neq \emptyset$.
(2) *If* $\Omega_a^+ \neq \emptyset$ *then* $\mathcal{N}^+ \cap \mathcal{E}^+ \neq \emptyset$.
(3) *If* $\Gamma_b^+ \neq \emptyset$ *then* $\mathcal{N}^- \cap \mathcal{E}^+ \neq \emptyset$.
(4) *If* $\Omega_a^- \neq \emptyset$ *and* $\Gamma_b^- \neq \emptyset$ *then* $\mathcal{N}^+ \cap \mathcal{A}^- \neq \emptyset$ *and* $\mathcal{N}^- \cap \mathcal{B}^- \neq \emptyset$.

Proof. (2) We can easily construct a $C^\infty \cap W_0^{1,p}(\Omega)$ function ξ with support on a small ball where $a > 0$ such that $E(\xi) > 0$.

(3) Let $v \in C^\infty$ be a function defined in $\partial\Omega$ such that $B(v) > 0$ and let $u \in W^{1,p}(\Omega)$ having v as its trace. Let ξ be a C^∞ cut function such that $0 \le \xi \le 1$, $\xi \equiv 1$ in a small ball $B(x_0, r) \subset \Omega$ where $|u| > c$ for some $c > 0$, $\xi \equiv 1$ in a neighbourhood $\Omega_\delta := \{x \in \overline{\Omega}\ ;\ dist\,(x, \partial\Omega) < \delta\}$ of $\partial\Omega$, and $\xi = 0$ in $\Omega \setminus \big(B(x, 2r) \cup \Omega_{2\delta}\big)$. We can assume that

$$c^p \int_{B(x_0,2r)} |\nabla\xi|^p \ge \int_{B(x_0,2r)} |\xi|^p(|\nabla u|^p - V|u|^p)$$

which implies that $E(\xi u) > 0$. Thus $\xi u \in \mathcal{B}^+ \cap \mathcal{E}^+$ and from Proposition 2.4 (i) we infer that $\mathcal{N}^+ \cap \mathcal{E}^+ \neq \emptyset$.

(4) We prove that $\Lambda^- \neq \emptyset$. Let $0 \leq v \in C^\infty$ be a function defined in $\partial\Omega$ such that $B(v) < 0$ and let $u \in W^{1,p}(\Omega)$ having v as its trace. We can assume that $u \geq 0$ by replacing u by $|u|$ if necessary. If $A(u) \geq 0$ let us take a function $0 \leq w \in W_0^{1,p}(\Omega)$ with support in Ω_a^- such that

$$\int_{\Omega_a^-} a w^r < -\int_\Omega a u^r$$

which implies that $A(u+w) < 0$. Hence replace u by $u+w$, which also has v as trace. Let ξ be a C^∞ cut function such that $0 \leq \xi \leq 1$, $\xi \equiv 1$ in a small ball $B(x_0, r) \subset \Omega_a^-$ where $|u| > c$ for some $c > 0$, $\xi \equiv 1$ in a neighbourhood $\Omega_\delta := \{x \in \overline{\Omega}\ ;\ dist\,(x, \partial\Omega) < \delta\}$ of $\partial\Omega$, and $\xi = 0$ in $\Omega \setminus \big(B(x, 2r) \cup \Omega_{2\delta}\big)$. We can assume that

$$c^p \int_{B(x_0,2r)} |\nabla \xi|^p \geq \int_{B(x_0,2r)} |\xi|^p (|\nabla u|^p - V|u|^p)$$

which implies that $E(\xi u) > 0$. The cut function ξ can be chosen to satisfy also

$$A(\xi u) > -\left(\frac{q-p}{q-r}\right)\left(\frac{p-r}{q-r}\right)^{\frac{p-r}{q-p}} \frac{(-E(\xi u))^{\frac{q-r}{q-p}}}{(-B(\xi u))^{\frac{p-r}{q-p}}} = \min_{t>0} m_{\xi u}(t).$$

Thus $\xi u \in \Lambda^-$. The proof of the other case is similar. □

We keep here the same notation for the different coerciveness constants, although they read now as follows

$$j_0(a) := \inf\left\{\int_\Omega |\nabla u|^p + V(x)|u|^p\ ;\ \int_\Omega a|u|^r = 0, \|u\|_{1,p} = 1\right\},$$

$$j_0(b) := \inf\left\{\int_\Omega |\nabla u|^p + V(x)|u|^p\ ;\ \int_{\partial\Omega} b|u|^q = 0, \|u\|_{1,p} = 1\right\}.$$

We left to the reader the expressions of $\mu^\pm$ in terms of the operators E, A, B. We can now formulate an existence and multiplicity result for Problem II. We generalize some results of [10,17] where this problem was studied for $V \equiv 0$, $a \equiv b \equiv 1$.

Theorem 6.7. (i) *Assume that $\Omega_a^+ \neq \emptyset$ and $\Gamma_b^+ \neq \emptyset$. If either $\lambda_1 > 0$ or $\lambda_1 = 0$ and $\int_\Omega a\varphi_1^r < 0$, $\int_{\partial\Omega} b\varphi_1^q < 0$ then there exists at least two solutions for any $\lambda \in (0, \mu_+^{-1})$.*

(ii) *Assume that $\Omega_a^+ \neq \emptyset$. If $\lambda_1 = 0$ and $\int_\Omega a\varphi_1^r < 0$ then there exists at least one solution for any $\lambda \in (0, \mu_+^{-1})$.*

(iii) *Assume that $\Omega_a^- \neq \emptyset$ and $\Gamma_b^- \neq \emptyset$. If $j_0(a) > 0$ and $j_0(b) > 0$ then there exist at least two solutions in $\mathcal{E}^-$ for any $\lambda \in (0, \mu_-^{-1})$.*

(iv) *Assume that $\Omega_a^+ \neq \emptyset$. If $\lambda_1 < 0 < \lambda_2$, $j_0(a) > 0$ and $\int_\Omega a\varphi_1^r < 0$ then there exist at least one solution in $\mathcal{E}^+$ for any $\lambda \in (0, \mu_+^{-1})$*

(v) *Assume that* $\Omega_a^+ \neq \emptyset$ *and* $\Gamma_b^+ \neq \emptyset$. *If* $\lambda_1 < 0 < \lambda_2$, $j_0(a) > 0$, $j_0(b) > 0$, $\int_\Omega a\varphi_1^r < 0$, *and* $\int_{\partial\Omega} b\varphi_1^q < 0$ *then there exist at least two solutions in* $\mathcal{E}^+$ *for any* $\lambda \in (0, \mu_+^{-1})$.

The proof of this theorem comes from Theorem 5.7 and the regularity results quoted in the proof of Theorem 6.2.

Corollary 6.8. *Assume that* $\Omega_a^\pm \neq \emptyset$ *and* $\Gamma_b^\pm \neq \emptyset$. *If* $\lambda_1 < 0 < \lambda_2$, $j_0(a) > 0$, $j_0(b) > 0$, $\int_\Omega a\varphi_1^r < 0$, *and* $\int_{\partial\Omega} b\varphi_1^q < 0$ *then there exists* **at least 4 solutions** *of Problem II for any* $\lambda \in (0, \min\{\mu_+)^{-1}, \mu_-^{-1}\})$.

Remark 6.9. In [10,17] the authors also proved non-existence of solutions for large values of λ. To our knowledge this is an open problem in the non coercive case.

6.3. Existence and multiplicity results for Problem III

Let us now discuss the solvability of

$$\begin{cases} \Delta_p^2 u - c|u|^{p-2}u = \lambda a(x)|u|^{r-2}u + b(x)|u|^{q-2}u & \text{in } \Omega; \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases}$$

The open bounded set $\Omega \subset \mathbb{R}^N$ is assumed here to have a Lipschitz boundary and $a, b, \in L^\infty(\Omega)$ and $c \in \mathbb{R}$. By a solution of this problem we mean a function $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ such that

$$\int_\Omega |\Delta u|^{p-2}\Delta u\, \Delta\varphi = \int_\Omega \left(c|u|^{p-2}u + \lambda a(x)|u|^{r-2}u + b(x)|u|^{q-2}u\right)\varphi \tag{6.7}$$

holds for all $\varphi \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. Here $V, a, b \in L^\infty(\Omega)$ and $1 < r < p < q < p^{**}$, where

$$p^{**} := \frac{pN}{N-2p} \text{ if } p < \frac{N}{2}; \quad p^{**} := +\infty \text{ otherwise},$$

is the critical Sobolev exponent for $W^{2,p}(\Omega)$. The space $W(\Omega) := W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ endowed with the equivalent norm

$$\|u\|_W := \|\Delta u\|_p$$

is reflexive uniformly convex Banach space. Take $X = W(\Omega)$,

$$\begin{aligned} E(u) = \int_\Omega (|\Delta u|^p - c|u|^p)\, dx, \quad A(u) \\ = \lambda \int_\Omega a(x)|u|^r\, dx, \quad B(u) = \int_\Omega b(x)|u|^q\, dx, \end{aligned}$$

and the energy functional

$$\begin{aligned} J(u) &= \frac{1}{p}E(u) - \frac{1}{r}A(u) - \frac{1}{q}B(u) \\ &= \frac{1}{p}\int_\Omega (|\Delta u|^p - c|u|^p)\, dx - \frac{\lambda}{r}\int_\Omega a(x)|u|^r\, dx - \frac{1}{q}\int_\Omega b(x)|u|^q\, d\rho. \end{aligned}$$

It is clear that A, B satisfy (2.1) and hypothesis (**H1**) since the embedding $W(\Omega) \hookrightarrow L^r(\Omega)$ and $W(\Omega) \hookrightarrow L^q(\Omega)$ are compact (because $r, q < p^{**}$). Thus (**H1**), (**H2**) and (**H3**) hold.

The coerciveness of E depends on the position of c with respect to the *first eigenvalue of the following* $p-$*bilaplacian* problem

$$\begin{cases} \Delta_p^2 u = \mu |u|^{p-1} u & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.8}$$

with weak formulation

$$\int_\Omega |\Delta u|^{p-2} \Delta u \Delta \varphi \, dx = \mu \int_\Omega |u|^{p-2} u \varphi \, dx \quad \forall \varphi \in W(\Omega).$$

According to [9] there exists $\lambda_1 > 0$ a least eigenvalue of (6.8), and this eigenvalue is principal, isolated and simple. Furthermore it holds

$$\lambda_1 = \inf \left\{ \int_\Omega |\Delta u|^p \; ; \; u \in W(\Omega), \|u\|_p = 1 \right\}.$$

Let $\varphi_1 > 0$ with $\|\varphi_1\|_p = 1$ be an eigenfunction associated to λ_1. Denote by

$$\lambda_2 = \inf\{\lambda > \lambda_1 \; ; \; \lambda \text{ solves } (6.8)\}.$$

The fact that $\lambda_2 \in \mathbb{R}$ is an eigenvalue of the Dirichlet p-bilaplacian and the existence of a sequence of eigenvalues is proved for instance in [19]. Finally let us denote

$$\Omega_a^\pm := \{x \in \Omega \; ; \; a(x) \gtrless 0\}, \quad \Omega_b^\pm := \{x \in \Omega \; ; \; b(x) \gtrless 0\},$$
$$\Omega_{a,0} := \{x \in \Omega \; ; \; a(x) = 0\}, \quad \Omega_{b,0} := \{x \in \Omega \; ; \; b(x) = 0\}.$$

Lemma 6.10. (1) *$\mathcal{E}^+ \neq \emptyset$.*
(2) *If $\Omega_a^+ \neq \emptyset$ then $\mathcal{N}^+ \cap \mathcal{E}^+ \neq \emptyset$.*
(3) *If $\Omega_b^+ \neq \emptyset$ then $\mathcal{N}^- \cap \mathcal{E}^+ \neq \emptyset$.*
(4) *If $\Omega_a^- \cap \Omega_b^- \neq \emptyset$ then $\mathcal{N}^+ \cap \mathcal{A}^- \neq \emptyset$ and $\mathcal{N}^- \cap \mathcal{B}^- \neq \emptyset$.*

Proof. (2) We distinguish two cases:

a) $\Omega_a^+ \cap (\Omega_b^- \cup \Omega_{b,0}) \neq \emptyset$. In this case we can construct a $C^\infty \cap W(\Omega)$ function v with support in a small ball $B(x_0, 2r) \subset \Omega_a^+ \cap (\Omega_b^- \cup \Omega_{b,0})$ such that $A(v) > 0 \geq B(v)$. Let ξ be a C^∞ cut function such that $0 \leq \xi \leq 1$, $\xi \equiv 1$ in $B(x_0, r) \subset \Omega$ where $|v| > \epsilon$ for some $\epsilon > 0$, $\xi \equiv 0$ in $\overline{\Omega} \setminus B(x_0, 2r)$. We can assume that

$$\epsilon^p \int_{B(x_0,2r)} |\Delta \xi|^p \geq \int_{B(x_0,2r)} |\xi|^p (|\Delta v|^p + c|v|^p) - 2 \int_\Omega |\nabla v|^p |\nabla \xi|^p \tag{6.9}$$

which implies that $E(\xi v) > 0$. Thus $\xi u \in \mathcal{A}^+ \cap \mathcal{B}_0^- \cap \mathcal{E}^+$ and from Proposition 2.4 (i) we infer that $\mathcal{N}^+ \cap \mathcal{E}^+ \neq \emptyset$.

b) If $\Omega_a^+ \subset \Omega_b^+$ the construction of v and ξ is analogous to the previous case, starting with $v \in C^\infty \Omega)$ such that $A(v), B(v) > 0$ with support in a small ball $B(x_0, 2r) \subset \Omega_a^+$. The cut function ξ can be chosen in such a way that (6.9) is satisfied as well as

$$A(\xi v) < \left(\frac{q-p}{q-r}\right) \left(\frac{p-r}{q-r}\right)^{\frac{p-r}{q-p}} \frac{E(\xi v)^{\frac{q-r}{q-p}}}{B(\xi v)^{\frac{p-r}{q-p}}} = \max_{t>0} m_{\xi v}(t).$$

The proof of the other cases are similar.

□

The coerciveness constants are now

$$j_0(a) := \inf\left\{\int_\Omega |\Delta u|^p - c|u|^p \ ; \ \int_\Omega a|u|^r = 0, \|u\|_W = 1\right\},$$
$$j_0(b) := \inf\left\{\int_\Omega |\Delta u|^p - c|u|^p \ ; \ \int_\Omega b|u|^q = 0, \|u\|_W = 1\right\}$$

and one also has to rewrite the constants $\mu^\pm$ in terms of the operators E, A, B. By applying Theorem 5.7 we get

Theorem 6.11. (i) *Assume that $\Omega_a^+ \neq \emptyset$ and $\Omega_b^+ \neq \emptyset$. If either $c < \lambda_1$ or $c = \lambda_1$ and $\int_\Omega a\varphi_1^r < 0$, $\int_\Omega b\varphi_1^q < 0$ then there exist at least two solutions for any $\lambda \in (0, \mu_+^{-1})$.*

(ii) *Assume that $\Omega_a^+ \neq \emptyset$. If $c = \lambda_1$ and $\int_\Omega a\varphi_1^r < 0$ then there exists at least one solution for any $\lambda \in (0, \mu_+^{-1})$.*

(iii) *Assume that $\Omega_a^- \cap \Omega_b^- \neq \emptyset$. If $j_0(a) > 0$ and $j_0(b) > 0$ then there exist at least two solutions in $\mathcal{E}^-$ for any $\lambda \in (0, \mu_-^{-1})$.*

(iv) *Assume that $\Omega_a^+ \neq \emptyset$. If $\lambda_1 < c < \lambda_2$, $j_0(a) > 0$ and $\int_\Omega a\varphi_1^r < 0$ then there exists at least one solution in $\mathcal{E}^+$ for any $\lambda \in (0, \mu_+^{-1})$*

(v) *Assume that $\Omega_a^+ \neq \emptyset$ and $\Omega_b^+ \neq \emptyset$. If $\lambda_1 < c < \lambda_2$, $j_0(a) > 0$, $j_0(b) > 0$, $\int_\Omega a\varphi_1^r < 0$, and $\int_\Omega b\varphi_1^q < 0$ then there exist at least two solutions in $\mathcal{E}^+$ for any $\lambda \in (0, \mu_+^{-1})$.*

Corollary 6.12. *Assume that $\Omega_a^+ \neq \emptyset$, $\Omega_b^+ \neq \emptyset$ and $\Omega_a^- \cap \Omega_b^- \neq \emptyset$. If $\lambda_1 < c < \lambda_2$, $j_0(a) > 0$, $j_0(b) > 0$, $\int_\Omega a\varphi_1^r < 0$, and $\int_\Omega b\varphi_1^q < 0$ then there exist* **at least 4 solutions** *for all $\lambda \in (0, \min\{\mu_+)^{-1}, \mu_-^{-1}\})$.*

Remark 6.13. Notice that we do not claim in the statement of Theorem 6.11 that the solutions are positive. The reason is that we can not use, for a local minimizer $u \in W^{2,p}(\Omega)$, the relation $J(u) = J(|u|)$ to deduce positivity of solution, since it can happen that $|u| \notin W^{2,p}(\Omega)$ even if $u \in W^{2,p}(\Omega)$. The existence of positive solutions of Problem III is, to our knowledge, an open problem.

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